1. Basic intro to singular chain complexes, compute homology of a point.

   (a) Basic understanding of simplices, give definition.

   Definition. Δ_n, *the* n*-simplex, is defined as*

   $$\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1, x_i \geq 0\}.$$

   Definition. *The* i*th* ***face map*** *is a map*

   $$F_i : \Delta_n \to \Delta_{n+1},$$

   given by adding in a zero in any possible position.

   From these, we get a complex $S_*(X)$, the singular complex of X. It's objects are

   $$S_n(X) := \mathbb{Z}[\mathrm{Hom}(\Delta_n, X)]$$

   which we call the n-chains in X.
   Draw pictures
   Each F_i induces a map $F_i^* : \mathrm{Hom}(\Delta_{n+1}, X) \to \mathrm{Hom}(\Delta_n, X)$.
   Collecting these together gives a map

   $$\partial_{n+1} : S_{n+1}(X) \to S_n(X)$$
   $$\partial_{n+1} = \sum_{i=0}^{n+1} F_i^*$$

   Called the **boundary map**
   An easy computation yields that this is a complex, $S_*(X)$. It's homology groups are $H_*(X)$ are topological and homotopy invariants, the homology of X.

   (b) Compute homology of a point. Let $\sigma_n : \Delta^n \to *$ be the unique map. Let f_i be the inclusion of the ith face. Then

   $$S_n(X) \cong \mathbb{Z}$$

   with σ_n as basis. We compute

   $$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_n \circ f_i = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sigma_{n-1} & \text{if } n \text{ is even} \end{cases}$$

So the singular complex is

$$\dots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

With homologies

$$\dots \qquad 0 \qquad 0 \qquad 0 \qquad \mathbb{Z}$$

2. Basic covering space stuff

   (a) Define a covering map/space.

   Definition. *Let $\varphi : E \to X$ be a continuous map. Then we say φ is a covering map if every point $p \in X$ has an evenly covered neighborhood U.*
   This means that $\varphi^{-1}(U) = \coprod_{\alpha \in A} U_\alpha$ where $\varphi : U_\alpha \to U$ is a homeomorphism.

   Draw pictures.

   (b) Define classifying space, show BG is Eilenberg-Maclane (long exact sequence in homotopy)

   (c) Define a (proper) group action on a space.

   Definition. *A group action of G on a space X is a homomorphism*

$$\rho : G \to \{\text{Homeomorphisms} \quad X \to X\}$$

   Definition. *An action ρ is called* ***proper*** *if The resulting quotient*

$$X \to X/G$$

   is a covering map.

3. Prove that if G acts on X, then G acts on $S_*(X)$, with

$$S_*(X)_G \cong S_*(X/G)$$

   (Lemma 6.10.2)

Proof. We have an obvious map $S_n(X) \xrightarrow{\pi_*} S_n(X/G)$ given by composition with $X \xrightarrow{\pi} X/g$. Suppose we have $\sigma : \Delta_n \to X$ and $g \in G$. Then TFDC

$$\begin{array}{ccccc} \Delta_n & \xrightarrow{\sigma} & X & \xrightarrow{g} & X \\ & \searrow & \downarrow & \swarrow & \\ & & X/G & & \end{array}$$

Thus, π_* descends to the quotient $S_*(X)_G \to S_*(X/G)$. The unique lifting property of covering maps yields that this is an isomorphism.

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow \\ \Delta_n & \longrightarrow & X/G \end{array}$$

$\square$

Definition. *Let G be a group. Suppose we have a contractible space EG on which G acts properly. Then the quotient space $X/G =: BG$ is called a* ***Classifying space*** *for G.* ‘

4.

Theorem 1. *Let BG be a classifying space for G. Then the (co)homology of BG is naturally isomorphic to the (co)homology of G.*

5. Examples:

   (a) $\mathbb{S}^1$ as a classifying space for $\mathbb{R}$.

   (b) $\mathbb{RP}^\infty$ as classifying space for $\mathbb{Z}/2$

   (c) $B\mathbb{Z}/n$ gives finitely generated abelian groups.