Representation Theory Prelim Problems

2018, 7. Let G be a finite group and π, π' be two irreducible representations of G. Prove or disprove the following assertion: π and π' are equivalent if and only if $\det \pi(g) = \det \pi'(g)$ for all $g \in G$.

2017, 2. Show that a finite simple group cannot have a 2-dimensional irreducible representation over $\mathbb{C}$. (Hint: the determinant might prove useful.)

2015, 7. Let G be a non-abelian group of order p^3 with p a prime.

(a) Determine the order of the center Z of G.

(b) Determine the number of inequivalent complex 1-dimensional representations of G.

(c) Compute the dimensions of all the inequivalent irreducible representations of G and verify that the number of such representations equals the number of conjugacy classes of G.

2014, 7. Let C_p denote the cyclic group of order p.

(a) Show that C_p has two irreducible representations over $\mathbb{Q}$ (up to isomorphism), one of dimension 1 and one of dimension $p - 1$.

(b) Let G be a finite group, and let $\rho : G \to \mathrm{GL}_n(\mathbb{Q})$ be a representation of G over $\mathbb{Q}$. Let $\rho_{\mathbb{C}} : G \to \mathrm{GL}_n(\mathbb{C})$ denote ρ followed by the inclusion $\mathrm{GL}_n(\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$. Thus $\rho_{\mathbb{C}}$ is a representation of G over $\mathbb{C}$, called the *complexification* of ρ. We say that an irreducible representation ρ of G is *absolutely irreducible* if its complexification remains irreducible over $\mathbb{C}$.
Now suppose G is abelian and that every representation of G over $\mathbb{Q}$ is absolutely irreducible. Show that $G \cong (C_2)^k$ for some k (i.e., is a product of cyclic groups of order 2).

2014, 8. Let G be a finite group and $\mathbb{Z}[G]$ the internal group algebra. Let $\mathcal{Z}$ be the center of $\mathbb{Z}[G]$. For each conjugacy class $C \subseteq G$, let $P_C = \sum_{g \in C} g$.

(a) Show that the elements P_C form a $\mathbb{Z}$-basis for $\mathcal{Z}$. Hence $\mathcal{Z} \cong \mathbb{Z}^d$ as an abelian group, where d is the number of conjugacy classes in G.

(b) Show that if a ring R is isomorphic to $\mathbb{Z}^d$ as an abelian group, then every element in R satisfies a monic integral polynomial. (**Hint:** Let $\{v_1, \ldots, v_d\}$ be a basis of R and for a fixed non-zero $r \in R$, write $rv_i = \sum_j a_{ij} v_j$. Use the Hamilton-Cayley theorem.)

(c) Let $\pi : G \to \mathrm{GL}(V)$ be an irreducible representation of G (over $\mathbb{C}$). Show that $\pi(P_C)$ acts on V as multiplication by the scalar

$$\frac{|C|\chi_\pi(C)}{\dim V},$$

where $\chi_\pi(C)$ is the value of the character χ_π on any element of C.

(d) Conclude that $|C|\chi_\pi(C)/\dim V$ is an algebraic integer.

2011, 6. Given a finite group G, recall that its *regular representation* is the representation on the complex group algebra $\mathbb{C}[G]$ induced by left multiplication of G on itself and its *adjoint representation* is the representation on the complex group algebra $\mathbb{C}[G]$ induced by conjugation of G on itself.

(a) Let $G = \mathrm{GL}_2(\mathbb{F}_2)$. Describe the number and dimensions of the irreducible representations of G. Then describe the decomposition of its regular representation as a direct sum of irreducible representations.

(b) Let G be a group of order 12. Show that its adjoint representation is reducible; that is, there is an H-invariant subspace of $\mathbb{C}[H]$ besides 0 and $\mathbb{C}[H]$.

2010, 3. Let $\mathbb{F}$ be a field of characteristic p, and G a group of order p^n. Let $R = \mathbb{F}[G]$ be the group ring (group algebra) of G over $\mathbb{F}$, and let $u := \sum_{x \in G} x$ (so u is an element of R).

(a) Prove that u lies in the center of R.

(b) Verify that Ru is a 2-sided ideal of R.

(c) Show there exists a positive integer k such that $u^k = 0$. Conclude that for such a k, $(Ru)^k = 0$.

(d) Show that R is **not** a semi-simple ring. (**Warning:** Please use the definition of a semi-simple ring: do **not** use the result that a finite length ring fails to be semisimple if and only if it has a non-zero nilpotent ideal.)

2010, 8. Let G be the unique non-abelian group of order 21.

(a) Describe all 1-dimensional complex representations of G.

(b) How many (non-isomorphic) irreducible complex representations does G have and what are their dimensions?

(c) Determine the character table of G.

2009, 7. Let G be a finite group, k an algebraically closed field, and V an irreducible k-linear representation of G.

(a) Show that $\mathrm{Hom}_{kG}(V, V)$ is a division algebra with k in its center.

(b) Show that V is finite-dimensional over k, and conclude that $\mathrm{Hom}_{kG}(V, V)$ is also finite dimensional.

(c) Show the inclusion $k \hookrightarrow \mathrm{Hom}_{kG}(V, V)$ found in (a) is an isomorphism. (For $f \in \mathrm{Hom}_{kG}(V, V)$, view f as a linear transformation and consider $f - \alpha I$, where α is an eigenvalue of f).

2008, 7.

(a) Let G be a group of (finite) order n. Show that any irreducible left module over the group algebra $\mathbb{C}G$ has complex dimension at least $\sqrt{n}$.

(b) Give an example of a group G of order $n \geq 5$ and an irreducible left module over $\mathbb{C}G$ of complex dimension $\lfloor \sqrt{n} \rfloor$, the greatest integer to $\sqrt{n}$.

2006, 4. Let K/F be a finite Galois extension and let $n = [K : F]$. There is a theorem (often referred to as the "normal basis theorem") which states that there exists an irreducible polynomial $f(x) \in F[x]$ whose roots form a basis for K as a vector space over F. You may assume that theorem in this problem.

(a) Let $G = \mathrm{Gal}(K/F)$. The action of G on K makes K into a finite-dimensional representation space for G over F. Prove that K is isomorphic to the regular representation for G over F. (The regular representation is defined by letting G act on the group algebra $F[G]$ by multiplication on the left.)

(b) Suppose that the Galois group G is cyclic and that F contains a primitive n^{th} root of unity. Show that there exists an injective homomorphism $\chi : G \to F^{\times}$.

(c) Show that K contains a non-zero element a with the following property:

$$g(a) = \chi(g) \cdot a$$

for all $g \in G$.

(d) If a has the property stated in (c), show that $K = F(a)$ and that $a^n \in F^{\times}$.