Homework 8 for 506, Spring 2019
due Friday, June 7

Throughout this homework, A is a commutative ring with identity.

Problem 1. Let M be an A-module. Prove that the following are equivalent.

(1) $M \cong 0$
(2) $M_{\mathfrak{p}} \cong 0$ for any prime ideal $\mathfrak{p}$ of A
(3) $M_{\mathfrak{m}} \cong 0$ for any maximal ideal $\mathfrak{m}$ of A

Problem 2. Snake lemma. Assume that in the following commutative diagram of A-modules, the rows are exact:

$$\begin{array}{ccccccccc} & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N''. & & \end{array}$$

Show that there is an exact sequence:

$$\operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h$$

Problem 3. Show that the following are equivalent:

(1) P is a projective A-module (as defined in HW 7 for 505. Feel free to use problem 5 from that homework);
(2) Any surjective A-module homomorphism $M \to P$ splits;
(3) $\operatorname{Hom}_A(P, -)$ is an exact functor.

Problem 4. Let I be an A-module. Prove that the following are equivalent:

(1) For any injective homomorphism $i : M' \to M$ and any homomorphism $g : M' \to I$ there exists $h : M \to I$ such that the following diagram commutes:

(1)
$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \stackrel{i}{\rightarrowtail} & M \\ & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle h} & \\ & & I & & \end{array}$$

(2) The functor $\operatorname{Hom}_A(-, I) : \underline{\underline{A-\text{mod}}} \to \underline{\underline{A-\text{mod}}}$ is exact
(3) Any exact sequence $0 \to I \to M \to M'' \to 0$ splits

Hint: to prove 3) implies 1), take any diagram as in 1) and consider the module

$$I \oplus_{M'} M \stackrel{def}{=} \frac{I \oplus M}{\{(g(m'), -i(m')) \mid m' \in M'\}},$$

which is the *push-out* of the diagram (1):

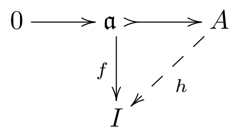
$$\begin{array}{ccc} M' & \stackrel{i}{\rightarrowtail} & M \\ \downarrow{\scriptstyle f} & & \downarrow \\ I & \longrightarrow & I \oplus_{M'} M \end{array}$$

Show that the bottom horizontal map (induced by i) is injective; then apply 3) to that map.

Definition. A module satisfying one of these conditions is called **injective**.

In the next problem we shall describe injective modules over $\mathbb{Z}$. Note that this is a bit more involved than describing projective modules which are just $\mathbb{Z}^n$.

Problem 5.
I. Prove the **Baer's criterion** for injective modules: An A-module I is injective if and only if for any ideal $\mathfrak{a} \subset A$ and any map $f : \mathfrak{a} \to I$, the map f can be extended to $h : A \to I$:



II. Show that an abelian group is injective (as a $\mathbb{Z}$-module) if an only if it is divisible. (An abelian group A is divisible if for any $a \in A$, and any $n \in \mathbb{Z}$ there exists $b \in A$ such that $a = nb$. For example, $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ are divisible.)