Homework 7 for 506, Spring 2019
due Friday, May 31

A is a commutative ring with identity.

Problem 1. Let $\mathfrak{a} \subset A$ be an ideal, and M be an A-module. Show that

$$A/\mathfrak{a} \otimes_A M \simeq M/\mathfrak{a}M.$$

Problem 2. Prove that the Hom-functor is left exact, and even more:

(1) Show that $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of A-modules if and only if for any A-module N,

$$0 \longrightarrow \operatorname{Hom}(M'', N) \longrightarrow \operatorname{Hom}(M, N) \longrightarrow \operatorname{Hom}(M', N)$$

is exact;

(2) Show that $0 \longrightarrow N' \longrightarrow N \longrightarrow N''$ is a an exact sequence of A-modules if and only if for any A-module M

$$0 \longrightarrow \operatorname{Hom}(M, N') \longrightarrow \operatorname{Hom}(M, N) \longrightarrow \operatorname{Hom}(M, N'')$$

is exact.

For both cases give examples showing that Hom is not exact.

Problem 3. Calculate:

(1) $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$
(2) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$

Problem 4. Let M, N, P be A-modules. Show that there is a canonical isomorphism:

$$(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$$

Problem 5.
Let $f \in A$, and let $X = \operatorname{Spec} A$. The canonical homomorphism $\phi : A \to A_f$ induces a continuous map $\phi^* : \operatorname{Spec} A_f \to X = \operatorname{Spec} A$. Show that ϕ^* induces a homeomorphism between $\operatorname{Spec} A_f$ and the principal open set X_f.