WORKSHEET ON ARTINIAN RINGS

DUE WEDNESDAY, MAY 15TH

All rings are commutative with 1 unless specified otherwise. This worksheet pursues two main results on Artinian rings:

(1) An Artinian ring is a Noetherian ring of dimension 0 (Thm 9).
(2) Structure theorem for Artinian rings (Thm 12)

Definition 1. An ideal I is nilpotent if $I^n = \langle x_1 \cdots x_n \,|\, x_i \in I \rangle = (0)$ for some $n \in \mathbb{Z}$.

Lemma 2. *Let R be a commutative Artinian ring. Show that $J(R)$ is a nilpotent ideal.*

Proof. Refer to Homework 2, problem 4. □

Theorem 3 (Hopkins-Levitzki theorem)**.** (1) *Let R be an Artinian ring, and M be a finitely generated R-module. Prove that M is a Noetherian R-module.*
(2) *Conclude that an Artinian ring is Noetherian.*

10pts. Feel free to cut and paste your proof from Homework 2 if you'd done it then. □

Lemma 4. *Let R be an Artinian integral domain. Then R is a field.*

5pts. □

Proposition 5. *Let R be an Artinian ring. Then any prime ideal is maximal.*

5pts. □

Corollary 6. *Let R be an Artinian ring. Then the Krull dimension of R is zero.*

1pt. □

Proposition 7. *Let R be a Noetherian ring. Then $\mathfrak{N}(R)$ is a nilpotent ideal.*

5pts. □

Lemma 8. *Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals in R which are also relatively prime. Then $\mathfrak{p}_1^n, \mathfrak{p}_2^m$ are relatively prime for any $n, m > 0$.*

5pts. (use properties of radicals from Homework 3) □

Theorem 9. *A ring R is Artinian if and only if it is Noetherian of Krull dimension 0.*

10pts. □

Corollary 10. *Let R be a Noetherian local ring with a maximal ideal $\mathfrak{m}$. Then one of the following holds:*
(•) *either $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for any $n > 0$*
(•) *or there exists n such that $\mathfrak{m}^n = 0$. In the latter case, R is Artinian.*

10pts. □

In other words, a local Noetherian ring is Artinian if and only if the unique maximal ideal is nilpotent.

Lemma 11. *Let R_1, R_2 be Artinian rings. Then $R_1 \times R_2$ is also Artinian.*

5pts. □

Theorem 12. *Any Artinian ring is isomorphic to a direct product of finitely many local Artinian rings. Moreover, the factors in such a decomposition are unique up to isomorphism and reordering.*

10pts. □

Remark 13. For an Artinian ring R, $\operatorname{Spec} R$ is just a union of finitely many points. Zariski topology becomes a discrete topology. $\operatorname{Spec} R$ is irreducible if and only if R is local.