Homework 4 for 506, Spring 2019
due Wednesday, May 8
Preliminary version

Problem 1.
(1). Show that $\operatorname{Spec} R$ is irreducible if and only if the nilradical $\mathfrak{N}$ is prime.
(2). Show that if R is a Noetherian ring, then $\operatorname{Spec} R$ is a Noetherian topological space.

Note: the converse is NOT true. It is possible to construct a local ring such that the maximal ideal is also a unique prime ideal, but containing an infinite ascending chain of ideals. The following, though, does hold: If $\operatorname{Spec} R$ is Noetherian, then the set of prime ideals of R satisfies the a.c.c.

Problem 2. Let $X = \operatorname{Spec} R$. Show that the principal open set X_f is quasi-compact. (There is a hint in [AM], Ch. I, Ex. 17.)

Problem 3.

(1) Let $x \in \operatorname{Spec} R$ be a point corresponding to the prime ideal $\mathfrak{p}$. Show that the closure of x in Zariski topology equals $V(\mathfrak{p})$.
(2) Show that the point $x \in \operatorname{Spec} R$ corresponding to the prime ideal $\mathfrak{p}$ is closed if and only if $\mathfrak{p}$ is a maximal ideal
(3) Show that if $\mathfrak{p}$ is a prime ideal then $V(\mathfrak{p})$ is irreducible.