Homework 3 for 506, Spring 2019
due Wednesday, May 1

Defn. A prime ideal $\mathfrak{p} \in R$ is *minimal* if it does not contain any proper prime ideals.

Example. If R is an integral domain then the ONLY minimal prime is 0-ideal.

Problem 1. Let R be a Noetherian ring. Show that there is one-to-one correspondence between irreducible components of $\operatorname{Spec} R$ and minimal prime ideals of R.

Note that this trivially implies that a Noetherian ring has finitely many minimal prime ideals.

Problem 2. Describe $\operatorname{Spec} R$ for

(1) $R = \mathbb{R}[x]/(x^n)$ for some $n > 0$
(2) $R = \mathbb{Z}[x]$ (here, just describe the points of this space)

Creative pictures are appreciated!

Problem 3. Show that for a topological space X to be irreducible is equivalent to any of the following

(1) Any non-empty open subset is dense;
(2) Any two non-empty open subsets have a non-empty intersection.

Problem 4. Let X be a topological space. Prove the following:

(1) If $Y \subset X$ is irreducible, then $\overline{Y}$ is irreducible;
(2) Any irreducible subset is contained in a maximal irreducible subset.

Exercises from class.

Exercise 1: Radicals [10pts]. Prove the following:

(1) $\operatorname{rad}(\operatorname{rad}(\mathfrak{a})) = \operatorname{rad}(\mathfrak{a})$
(2) $\operatorname{rad}(\mathfrak{a}\mathfrak{b}) = \operatorname{rad}(\mathfrak{a} \cap \mathfrak{b}) = \operatorname{rad}(\mathfrak{a}) \cap \operatorname{rad}(\mathfrak{b})$
(3) $\operatorname{rad}(\mathfrak{a}) = (1) \Leftrightarrow \mathfrak{a} = (1)$
(4) $\operatorname{rad}(\mathfrak{a} + \mathfrak{b}) = \operatorname{rad}(\operatorname{rad}(\mathfrak{a}) + \operatorname{rad}(\mathfrak{b}))$
(5) $\operatorname{rad}(\mathfrak{p}^n) = \mathfrak{p}$