Homework 2 for 506, Spring 2019
due Wednesday, April 24

R is a commutative ring with 1 unless specified otherwise.

Problem 1. Show that a local ring does not have non-trivial idempotents.

The following problem should remind you of Gauss lemma.
Problem 2. Let $R[x]$ be a polynomial ring with coefficients in R, and let

$$f(x) = a_n x^n + \cdots + a_0 \in R[x].$$

Prove the following:

[i] f is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent;

[ii] f is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent.

Problem 3. Show that in $R[x]$ the nilradical coincides with the Jacobson radical.

Problem 4. Let R be a commutative Artinian ring. Show that $J(R)$ is a nilpotent ideal. (An ideal I is nilpotent if $I^n = \{x_1 \cdots x_n \,|\, x_i \in I\} = (0)$ for some $n \in \mathbb{Z}$.

The result of Problem 4 holds for non-commutative rings. It is likely that your proof would work without change, but you'd have to be more careful about multiplying from the left or from the right; you would also need to carefully reformulate statements from class about relationships between one-sided units and one-sided maximal ideals. For the next problem, feel free to assume that the result of Problem 4 holds for not necessarily commutative rings.

Problem 5 (Hopkins-Levitzki theorem) [Optional].
1). Let R be an Artinian ring (not necessarily commutative), and M be a finitely generated R-module. Prove that M is a Noetherian R-module.
2). Conclude that an Artinian ring is Noetherian.

Exercises from class.

Exercise 1 [5pts]. Show that if k is a field of characteristic p then the group algebra $k\mathbb{Z}/p$ is a local ring.

Exercise 2 [5pts]. Give an example of a ring R such that $R/\mathfrak{N}$ is not an integral domain.