Homework 1 for 506, Spring 2019
Due Wednesday, April 10
Preliminary version

A group algebra can be defined over any commutative ring.

Definition. Let R be a commutative ring, and G be a group. The group algebra RG is a free R-module with a basis $\{g\}_{g\in G}$, and multiplication defined on the basis elements as $g \times h = gh$, and extended linearly to RG.

Problem 1. Show that $\mathbb{Z}G$ is not semisimple.

The purpose of the next two problems is to illustrate that Maschke's theorem is "sharp": indeed, the statement could have been formulated as "if and only if".

Problem 2. Let p be a prime number. Classify isomorphism classes of indecomposable representations of the cyclic group $G = \mathbb{Z}/p$ over a field k of characteristic p.

Problem 3. Let p be a prime number. Prove the converse to Maschke's theorem: Let k be a field of characteristic p, and assume that G is a finite group of order divisible by p. Show that kG is not semisimple.

The next series of problems is about idempotent elements and their relationship to the decomposition of a semi-simple algebra as a product of simple algebras.

Definition. Let A be a ring.
(1) An element $e \in A$ is called an *idempotent* if $e^2 = e$.
(2) An idempotent e is called *central* if $e \in Z(A)$.
(3) Two idempotents e_1, e_2 are orthogonal if $e_1e_2 = 0$. A non-zero idempotent e is called *primitive* if there do not exist non-zero orthogonal idempotents $e_1, e_2 \in A$ such that $e = e_1 + e_2$.

Problem 4. Prove:

(1) An algebra is semisimple if and only if every left ideal has the form Ae for some idempotent $e \in A$.
(2) Let A be a semisimple algebra. Show that for any (two-sided) ideal I there exists a unique central idempotent $e \in A$ such that $I = Ae$. Conversely, if e is a central idempotent, then Ae is a two-sided ideal.
(3) Suppose $1 = e_1 + \ldots + e_n$, where $e_1, \ldots, e_n \in A$ are central orthogonal idempotents (that is, $e_ie_j = e_je_i = 0$ for $i \neq j$). Show that $A \cong Ae_1 \times \ldots \times Ae_n$.

Problem 5. Let A be a semisimple algebra, and $e \in A$ be an idempotent. Prove that the following are equivalent

(1) e is primitive.
(2) Ae is a minimal left ideal in A (= simple module).

(3) eAe is a division algebra.

Problem 6. Let A be a semisimple algebra, and let $A = A_1 \times A_2 \times \cdots \times A_n$ be a decomposition of A as a product of simple algebras.

(1) Prove that the (two-sided) ideals of A are of the form $\sum_J A_i$, where $J \subset \{1, \ldots, n\}$
(2) Show that there exist central orthogonal idempotents $\{e_1, \ldots, e_n\}$ such that $A_i = Ae_i$.
(3) Let M be an A-module, and let $M_i = e_i M$. Show that $M = M_1 \oplus \cdots \oplus M_n$