MATH 506 **Final exam** **due June 12, 2019, by 10:30am**

NAME: SIGNATURE:

Instructions: You may use class notes as well as homework and refer to anything proved or stated in class: make it clear which fact/theorem you are using. No other sources are allowed.

There are 6 problems on the test. You only need to solve 5 to get 100%. No partial credit will be given on the 6th problem. Please indicate very clearly which 5 problems you wish to be graded.

If any questions arise feel free to send Julia na e-mail. In particular, if you are unsure whether using a particular result is ok, ask!

Problem	Points	Credit
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	≤ 50	

All rings are commutative with identity.

Problem 1. Write down the character table for S_4 and justify it.

Problem 2. Let R be a local ring, and M, N be finitely generated R-modules. Show that if $M \otimes N = 0$ then $M = 0$ or $N = 0$.

Problem 3. Let p be a prime, and let $\mathbb{Z}_{(p)}$ be the localization at the prime ideal (p). Describe the following topological spaces (points, irreducible components and dimension):

(1) $\operatorname{Spec} \mathbb{Z}_{(p)}$,
(2) $\operatorname{Spec} \mathbb{Z}_{(p)}[x]$.

Problem 4. Let A be an integral domain and M be an A-module. We say that M is *torsion-free* if for any $a \in A$, $m \in M$, $am = 0$ implies $a = 0$ or $m = 0$.

Show that being *torsion-free* is a local property. That is, the following are equivalent:

(1) M is torsion-free.
(2) The $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is torsion-free for any prime ideal $\mathfrak{p}$ of A.
(3) The $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is torsion-free for any maximal ideal $\mathfrak{m}$ of A.

Problem 5. Let k be a field, and let $f, g \in k[x, y]$ be irreducible elements which do not divide each other. Show that $V((f, g))$ is a finite set.
You may use that the Krull dimension of $k[x, y]$ is 2.

Problem 6. Let k be a field (you may assume that k is infinite), and let R be a finitely generated k-algebra. Prove that R is an Artinian ring if and only if R is a finite k-algebra.

Hint for the only if direction: First, prove it for a local ring R with maximal ideal $\mathfrak{m}$. This can be done in two steps: show that $R/\mathfrak{m}$ is finite; then show that in a descending chain of ideals $\mathfrak{m} \supset \mathfrak{m}^2 \supset \ldots$ each quotient is finite.