Homework 0 for 506, Spring 2019
OPTIONAL. Due Wednesday, April 3
Preliminary version

For this homework, R is an algebra over a field k. All modules are assumed to be left modules.

Problem 1. Let

$$S = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and let $A \subset M_4(\mathbb{R})$ be the real subalgebra generated by S and T. Let $V = \mathbb{R}^4$, and let A act on V via matrix multiplication by S and T. Prove that

(1) V is a simple A-module.

(2) $\operatorname{End}_A(V) = \left\{ \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}, \quad a, b, c, d \in \mathbb{R} \right\}$

(3) $\operatorname{End}_A(V) \simeq \mathbb{H}$, the quaternion algebra over $\mathbb{R}$ (see, for example, Hw1 from the Fall quarter for the description of $\mathbb{H}$.)

Problem 2. Prove Lemma from class: $\operatorname{rad}(M/\operatorname{rad} M) = 0$.

Def. A simple ring is a ring which does not have proper non-trivial two-sided ideals.

Problem 3. Let D be a division ring. Show that $M_n(D)$ is a simple ring.
Remark: If $D = k$ is a field, then $M_n(k)$ is a **central simple algebra** over k. Namely, it is simple, and its center is the field k itself.

Problem 4. Let B be the subring of $M_n(k)$ consisting of upper-triangular matrices: all matrices with zero below the main diagonal. Show that $J(B)$, the Jacobson radical of B, is the set of strictly upper-triangular matrices: matrices with zeros on the main diagonal and below the main diagonal.

Problem 5. Let A_1, A_2 be rings and let L_1 (respectively, L_2) be a simple module for A_1 (respectively, A_2). Let $A = A_1 \times A_2$. Show that L_1, L_2 have natural structure of A-modules, and that $\operatorname{Hom}_A(L_1, L_2) = 0$

Reading assignment. Our first unit in the Spring will be on Representation Theory of finite groups. The main reference is the book by J.-P. Serre "Linear representations of finite groups". Recently published alternative is a book by Peter

Webb “A course in finite group representation theory”, with a version available on-line here: http://www-users.math.umn.edu/~webb/RepBook/RepBookLatex.pdf