Homework 4 for 504, Fall 2018
due Wednesday, October 24

Problem 1. Show that the dihedral group D_m of symmetries of regular m-gon is isomorphic to a subgoup of

(1) S_m,
(2) $\mathrm{GL}_2(\mathbb{C})$.

Note: We now know at least 4 different presentations of D_m: as a semi-direct product of cyclic groups, by generators and relations, a permutation representation, and a matrix representation.

Problem 2. Let G be a group. Prove that the following are equivalent:

(1) There exists a (finite) central series $\{e\} = G_0 < G_1 < \ldots < G_n = G$.
(2) The descending central series
$$\ldots < \Gamma_i = [\Gamma_{i-1}, G] < \Gamma_{i-1} < \ldots < \Gamma_1 < \Gamma_0 = G$$
terminates at $\Gamma_n = \{e\}$.
(3) The ascending central series $\{e\} = Z_0 < Z_1 < Z_2 \ldots$ (where $Z_i/Z_{i-1} = Z(G/Z_{i-1})$) terminates at $Z_n = G$.

Problem 3. Let $B_n < \mathrm{GL}_n(\mathbb{R})$ be the subgroup of upper-triangular matrices, $T_n < B_n$ be the subgroup of diagonal matrices, and $U_n < B_n$ be the subgroup of upper-triangular matrices with 1's on the main diagonal. Assume $n \geq 2$. Show that
(a) U_n is nilpotent. What is the minimal length of its central series?
(b) B_n is solvable but not nilpotent.
(c) B_n is isomorphic to a semi-direct product of T_n and U_n.

Note: U_n is called the unipotent subgroup, B_n - the Borel subgroup, and T_n is the torus (of $\mathrm{GL}_n(\mathbb{R})$). The statements are valid for any field of coefficients F, at least if characteristic is not 2, and so should be your proofs.

Definition. Let G be a p-group. The subgroup
$$\Phi(G) = \bigcap_{[G:H]=p} H$$
is called the **Frattini subgroup** of G.

Problems 4, 5. Prove the following statements:
(a). $\Phi(G) \lhd G$.

(b). $\Phi(G)$ is the minimal subgroup of G such that
$$G/\Phi(G) \simeq \mathbb{Z}/p \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$$
The group of the form $\mathbb{Z}/p \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ is called an *elementary abelian p-group.*

(c). $\Phi(G) = G^p[G,G]$ (where $G^p = \langle g^p \mid g \in G \rangle$).

(d). Let G' be another p-group and $f : G \to G'$ be a group homomorphism. Show that f is surjective if and only if the induced map

$$\bar{f} : G/\Phi(G) \to G'/\Phi(G')$$

is surjective (in particular, you need to check that the induced map is well-defined).

(e). $\{x_1, \ldots, x_n\}$ is a system of generators of G if and only if $\{\bar{x}_1, \ldots, \bar{x}_n\}$ is a system of generators of $G/\Phi(G)$.

(f). **Burnside Basis Theorem.** All minimal systems of generators for a finite p-group G have the same number of elements.

Note: Part (e) implies that $\Phi(G)$ is a *non-generator* subgroup. Namely, if we have any system of generators of G, we can throw out all elements in the system that belong to the Frattini subgroup, and still have a generating set. In other words, any minimal generating set does not contain elements from $\Phi(G)$. The Frattini subgroup $\Phi(G)$ can be defined for ANY group as an intersection of all maximal proper subgroups. It will still be normal and satisfy the "non-generating" property. But as we established in the very first homework, the Burnside basis theorem will not hold for general groups.