Homework 3 for 504, Fall 2018
due Wednesday, October 17

Problem 1. Let G be a p-group, and H be a non-trivial normal subgroup of G. Show that $H \cap Z(G) \neq 1$.

Problem 2. Let p, q be prime numbers.

(1) Show that $\mathrm{Aut}(\mathbb{Z}/qZ) \simeq \mathbb{Z}/(q-1)\mathbb{Z}$.
(2) Classify all groups of order pq.

Note: "To classify" means to establish a list of representatives of each isomorphism class, and to prove that your list is complete.

Problem 3. (1). Classify all finite groups of orders 1 through 10.
(2). Classify all groups of order 2015.

Problem 4. (Sylow subgroups of symmetric groups)
(1). What is the order of a Sylow 2-subgroup of S_{2^n}?
(2). Give an explicit description of a Sylow 2-subgroup of S_{2^n}.

Note: You can think how this would work for S_{p^n}, and, more generally, for any S_n.

Problem 5. Let $\mathbb{F}_p$ be a finite field of p elements.

(1). Find the order of the finite group $\mathrm{GL}_n(\mathbb{F}_p) \stackrel{\text{def}}{=} \{\text{invertible matrices over } \mathbb{F}_p\}$.
(2). Let $U_n(\mathbb{F}_p)$ be the subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ consisting of matrices which have 1's on the main diagonal, 0's below, and arbitrary entries (from $\mathbb{F}_p$) above. Show that it is a Sylow p-subgroup in $\mathrm{GL}_n(\mathbb{F}_p)$. Is it unique?
(3). Let A, B be two commuting $n \times n$ matrices over $\mathbb{F}_p$ of order p (that is, $A^p = B^p = \mathrm{Id}$ and $[A, B] = 0$). Show that there exists a matrix $S \in \mathrm{GL}_n(\mathbb{F}_p)$ such that both SAS^{-1} and SBS^{-1} are upper-triangular (in other words, A, B can be simultaneously conjugated into upper-triangular form).
(4). Is this still true if A, B do not commute?