Homework 2 for 504, Fall 2018
due Wednesday, October 10,

Problem 0. Prove that $\binom{p^a m}{p^a}$ is not divisible by p, a prime, if $(m, p) = 1$, and a is a positive integer.

Problem 1.

(1) Prove the Third Isomorphism Theorem (all three parts of it):
  (a) Let $H \triangleleft G$ be a normal subgroup of G. Show that there is one-to-one correspondence between subgroups of G containing H and subgroups of G/H
  (b) Show that normal subgroups correspond to normals subgroups under this correspondence
  (c) Let $K \triangleleft G, H \triangleleft G$ be *normal* subgroups of G fitting into a tower $K \triangleleft H \triangleleft G$. Show that there is an isomorphism
$$G/H \cong \frac{G/K}{H/K}.$$
(2) Give an example of a tower of groups $K \triangleleft H \triangleleft G$ such that K is not normal in G.

Problem 2. Let $H < G$ be a subgroup. Show that the normalizer $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ is the maximal subgroup of G in which H is normal.

Problem 3. Let H be a subgroup of G of index 2. Show that H is normal.

Problem 4. Let G be a finite group, and $H \triangleleft G$ be a normal subgroup. Suppose $(|H|, [G : H]) = 1$. Show that H is the ONLY subgroup of G of order $|H|$.

Definition. Let G be a group acting on the set X. The subgroup
$$G_x = \mathrm{Stab_G(x)} = \{\mathrm{g \in G \mid gx = x}\}$$
is called the **stabilizer** or the **isotropy group** of x.

Problem 5. (1). Consider the "standard" action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$ (that is, an $n \times n$ invertible matrix acts on an $n \times 1$ vector by matrix multiplication). Describe the orbits and isotropy groups (up to conjugation) of this action.
(2). Let $\mathrm{GL}_n(\mathbb{C})$ act on the set of all $n \times n$ complex matrices by conjugation. Determine the orbits.

Problem 6. Show that any group of order p^2 for p prime is abelian.