

On Compass and Straightedge Constructions: Means

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1 Introduction

Using a compass and straightedge is of central importance in Euclidean geometry, for Euclid based his first three postulates on figures that could only be constructed using these two tools. In short, geometric constructions were an absolute necessity for the ancient Greeks, in that Greek mathematicians could only affirm that a mathematical object existed only if they could construct the object in question. As a concrete example, one only needs to be reminded of Euclid's Proposition 1 ("To construct an equilateral triangle on a given finite straight line."). In order to show that an equilateral triangle actually existed, Euclid had to develop an algorithm to construct an equilateral triangle. For the duration of this paper, we will arm ourselves solely with a compass and straightedge and various theorems, lemmas, and corollaries from John Lee's *Axiomatic Geometry* [3].

2 Construction Ground Rules

As Lee carefully notes in *Axiomatic Geometry* [3], at the outset of our construction endeavors, we are allowed to use several operations that are based on Euclid's first three postulates:

- Given two distinct points A, B , construct the line segment \overline{AB} .
- Given a line segment in the plane, extend it in either direction to form a line segment longer than any predetermined length.
- Given two distinct points in the plane, construct the circle centered at the first point and passing through the second point.

Lee also makes sure to include several operations that are not explicitly found in the postulates but nonetheless are permissible in compass and straightedge constructions:

- Given two nonparallel lines, locate the point where they intersect.
- Give a circle and a secant line, locate the two points where they intersect.
- Given two nontangential circles, locate the points (if any) where they intersect.

3 Basic Constructions

3.1 Constructing an Equilateral Triangle

Given a segment \overline{AB} , we shall construct an equilateral triangle.

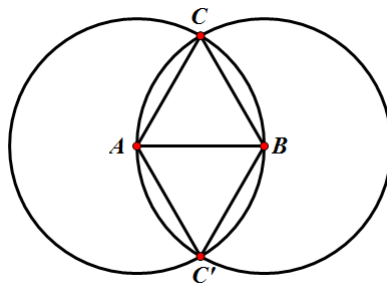


Figure 1: Construction of an Equilateral Triangle

1. Fastening one leg of our compass on A and placing our fastened pencil on B , construct a circle with center A and radius \overline{AB} .
2. Fastening one leg of our compass on B and placing our fastened pencil on A , construct a circle with center B and radius \overline{BA} .
3. Our two circles intersect at two points (one intersection point on each side of our segment \overline{AB}). Label these two intersection points C and C' .
4. With our straightedge, connect A and C to form the segment \overline{AC} .
5. Similarly, connect B and C to form the segment \overline{BC} .

$\triangle ABC$ is an equilateral triangle. Similarly, by repeating steps 4 and 5, $\triangle BAC'$ will also be an equilateral triangle.

Proof. By construction, \overline{AC} is a radius of the circle with radius \overline{AB} and center A . So \overline{AC} is congruent to \overline{AB} . Similarly, by construction, \overline{BC} is a radius of the circle with radius \overline{AB} and center B . So $\overline{BC} \cong \overline{AB}$. So transitively, $\overline{AC} \cong \overline{AB} \cong \overline{BC}$, so all three sides of $\triangle ABC$ are congruent, and therefore $\triangle ABC$ is an equilateral triangle. \square

3.2 Constructing the Midpoint of a Segment

Using our previous constructions of equilateral triangles $\triangle ABC$ and $\triangle BAC'$, we shall construct the midpoint of our given segment \overline{AB} .

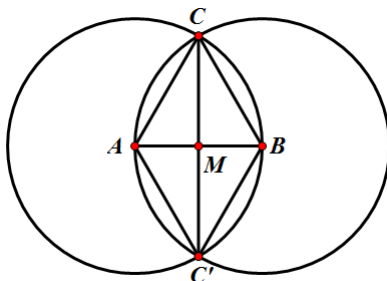


Figure 2: Construction of the Midpoint

1. With our straightedge, connect points C and C' to form segment $\overline{CC'}$. We now have triangles $\triangle C'CA$ and $\triangle CC'B$.
2. Since segments $\overline{CC'}$ and \overline{AB} intersect, label the point of intersection M .

The point M is the midpoint of segment \overline{AB} .

Proof. Segments \overline{AC} , \overline{BC} , $\overline{AC'}$, and $\overline{BC'}$ are all congruent to each other by transitivity since they are all radii of our constructed circles with centers A and B (whose radii are equal to each other by construction). The segment $\overline{CC'}$ is reflexively congruent to itself, so by SSS Congruence, $\triangle C'CA \cong \triangle CC'B$. Now, because $\triangle C'CA$ and $\triangle CC'B$ are isosceles triangles by definition, the Isosceles Triangle Theorem shows us that $\angle ACC' \cong \angle AC'C$ of $\triangle C'CA$ and $\angle BC'C \cong \angle BCC'$ of $\triangle CC'B$. By definition of congruent triangles and transitivity, we have $\angle ACC' \cong \angle BCC'$. So because we also have that \overline{CM} is reflexively congruent to itself, $\triangle CAM \cong \triangle CBM$ by Postulate 9 (The SAS

Postulate). Therefore, by definition of congruent triangles, we have $\overline{AM} \cong \overline{MB}$ and so $AM = MB$. Because $AM = MB$ and since $M \in \overline{AB}$, we have shown that M is indeed the midpoint of \overline{AB} by Lemma 3.26. \square

3.3 Constructing a Line Perpendicular to Another Line

Using our previous two constructions, we have constructed the segment \overline{CM} . \overline{CM} is perpendicular to our given segment \overline{AB} .

Proof. Because we have shown $\triangle ABC$ is equilateral in 3.1, $m\angle CAB = m\angle CBA = m\angle C = 60$ by the 60-60-60 Theorem. So because we have shown in 3.2 that $m\angle ACC' = m\angle BCC'$ and because $m\angle C = m\angle ACC' + m\angle BCC'$ by Theorem 4.8, we have $m\angle ACC' = m\angle BCC' = 30$. Now, since $m\angle CAB = m\angle CBA = 60$, the Angle-Sum Theorem for Triangles and simple algebra show us that $m\angle CMA = m\angle CMB = 90$. So by definition of perpendicular, segment \overline{CM} is perpendicular to segment \overline{AB} . In addition, segment \overline{CM} is also the perpendicular bisector of segment \overline{AB} since M is clearly contained in segment \overline{CM} by construction. \square

4 Three Means and Their Constructions

For the duration of this paper, we shall focus on exploring the arithmetic, geometric, and harmonic means. Before introducing each respective mean, however, we begin with stating the AM-GM-HM inequality:

$$AM \geq GM \geq HM.$$

This inequality will become illuminated once we construct each mean.

4.1 Arithmetic Mean

The arithmetic mean of two numbers a and b is $\frac{a+b}{2}$.

Example 4.1. George has completed the “Geometry for Teachers” sequence (Math 444, 445) at the UW. His final grades for the two courses were 3.7 and 2.1, respectively. (He explained to Julia that he had been rather busy during Math 445). Letting a denote 3.7 and b denote 2.1, the arithmetic mean of George’s two grades is $\frac{a+b}{2} = \frac{3.7+2.1}{2} = 2.9$.

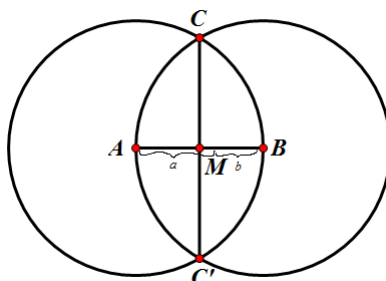


Figure 3: Construction of the Arithmetic Mean

Construction.

1. Draw a line segment such that the length is the sum of the two segments a and b .
2. Draw a circle with center A and radius $a + b$.
3. Draw another circle with center B and radius $a + b$.
4. Find the intersection points of the two circles and draw a line through them.
5. Find the point where that line intersects \overline{AB} . Call the point M .
6. The arithmetic mean is \overline{AM} or \overline{BM} . In both cases, the length of the arithmetic mean is $\frac{a+b}{2}$.

Justification. By Lemma 16.3, we can move segment b so that it is collinear with segment a , and such that they share an endpoint. Call the new combined segment \overline{AB} . Drawing circles is possible by Euclid's Postulate 3. The intersection points of two circles exist according to Theorem 14.10. By Corollary 2.27, we can draw a line through any two distinct points. By the Plane Separation Postulate, the intersection of the two lines and the point M exists.

Proof. The diameter of our circle is $|AB| = a + b$ since \overline{AB} is constructed by placing segment a and segment b end to end. Because the radius of a circle is $\frac{1}{2}$ the diameter, the radius of our circle is $\frac{a+b}{2}$. But this is, by definition, the arithmetic mean of our two numbers a and b . \square

4.2 Geometric Mean

The geometric mean of two numbers a and b is \sqrt{ab} .

Example 4.2. Suppose you have taken the Geometry for Teachers sequence and the class grade for the first quarter was out of 400 points. However, for the second quarter, the class grade was out of 40 points (Julia decided that grades should only be based on the final). Suppose your grades were $365/400$ and $25/40$ (It was a rough final). By employing the arithmetic mean, your average grade would be $\frac{365+25}{2} = 195$. This weighs the first score very heavily, and a change of 10 points in the first quarter would make a much larger difference than a change of 10 points in the second quarter. However, by taking the geometric mean, the bias of the range of each score is eliminated. So the geometric mean of the two test scores is $\sqrt{365 \times 25} = 95.5$.

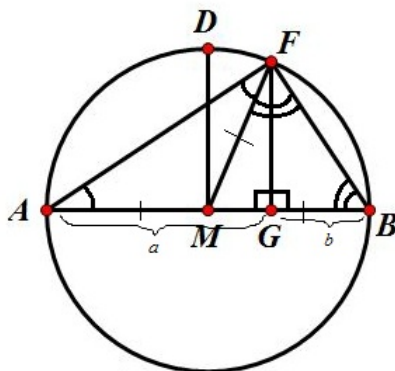


Figure 4: Construction of the Geometric Mean

Construction.

1. Construct the arithmetic mean through the same process in 4.1.
2. Call the point where the segments a and b meet the point G .
3. Using the compass, construct a circle centered at M with diameter AB .
4. From G , draw the line perpendicular to \overline{AB} . Call the intersection of this line and the semicircle the point F .
5. The length of \overline{GF} is the geometric mean of a and b .

Justification. By Euclid's Postulate 3, we can construct the circle centered at M with diameter AB . By Theorem 7.1, we can draw the line perpendicular to \overline{AB} .

Proof. Let us say $|GF| = g$, $|AG| = a$, and $|FB| = b$. Because \overline{GF} is perpendicular to \overline{AB} by construction, $\triangle AGF$ and $\triangle GFB$ are right triangles. Therefore, by Theorem 13.1, $|AF|^2 = |AG|^2 + |GF|^2$, so $|AF| = \sqrt{a^2 + g^2}$. Similarly, $|BF|^2 = |GB|^2 + |GF|^2$, so $|BF| = \sqrt{b^2 + g^2}$. Now, note that \overline{AM} is the radius of the circle centered at M , with diameter \overline{AB} . If we connect M and F , then \overline{MF} is also a radius of the circle. So $|AM| = |MF|$. Therefore $\triangle AMF$ is an isosceles triangle, so by Theorem 5.7, $\angle GAF \cong \angle GFA$. Also, \overline{MB} is a radius of the circle, so $|MB| = |MF|$. So again by Theorem 5.7, $\angle MBF \cong \angle BFM$. By Theorem 10.11, the angle sum of $\triangle AFB$ is 180° . So we have:

$$180^\circ = m\angle BAF + m\angle AFB + m\angle FBA,$$

$$180^\circ = 2m\angle BAF + 2m\angle FBA.$$

Also note that $m\angle AFB = m\angle BAF + m\angle FBA$. So $\angle AFB$ is equal to half the angle sum of $\triangle AFB$. So $m\angle AFB = 90^\circ$. So, using the Pythagorean Theorem (Theorem 13.1):

$$|AB|^2 = |AF|^2 + |FB|^2$$

$$(a + b)^2 = (\sqrt{a^2 + g^2})^2 + (\sqrt{b^2 + g^2})^2$$

$$a^2 + 2ab + b^2 = a^2 + 2g^2 + b^2$$

$$2ab = 2g^2$$

$$ab = g^2$$

$$\sqrt{ab} = g.$$

And thus, \overline{GF} is, by definition, the geometric mean. □

4.3 Harmonic Mean

The harmonic mean of two numbers a and b is $\frac{2}{\frac{1}{a} + \frac{1}{b}}$ or $\frac{2ab}{a+b}$.

Example 4.3. Because many students incorrectly apply the arithmetic mean to problems instead of the harmonic mean, a “story” problem is warranted for our discussion of the harmonic mean to illustrate its application.

Jim drives from New York City to Boston at a rate of 40 MPH and drives at a rate of 60 MPH on the return trip. What was his average speed for the entire trip?

A quick application of the arithmetic mean would yield an answer of 50 MPH ($\frac{40+60}{2} = 50$). However, 50 MPH is incorrect, for while the arithmetic mean is admittedly the most common type of average, it should not be employed when solving problems that deal with rates that are doing the same amount of work and contributing to the same “workload” (traveling one mile, for example). Put another way, each rate is contributing the same amount to the “output.” Half the result (distance traveled) stems from the first rate (40 MPH), and the other half stems from the second rate (60 MPH).

Let $X = 40$ MPH, $Y = 60$ MPH. X takes $\frac{1}{X}$ time, so it takes $\frac{1}{40}$ of an hour to travel one mile. Y takes $\frac{1}{Y}$ time, so it takes $\frac{1}{60}$ of an hour to travel one mile. So the total input is $\frac{1}{X} + \frac{1}{Y}$ for a total output of two miles.

So, in our particular example, our harmonic mean is $\frac{\text{Total Output}}{\text{Total Input}}$, which is $\frac{2}{\frac{1}{X} + \frac{1}{Y}} = \frac{2}{\frac{1}{40} + \frac{1}{60}} = 48$ MPH.

Our problem would be solved with the arithmetic mean if independence was brought into the driving scenario in the form of hours traveled for each rate. If the trip was split up into segments by means (no pun intended) of hours traveled for specific rates, we would no longer have rates working together for the same result.

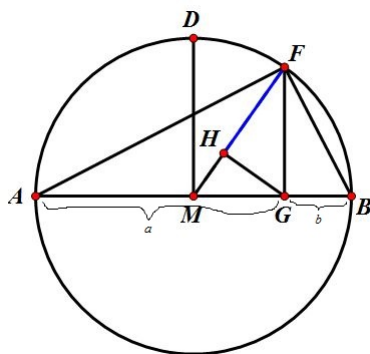


Figure 5: Construction of the Harmonic Mean

Construction.

1. Construct the arithmetic and geometric means through the same processes in 4.1 and 4.2.
2. Draw the line perpendicular to \overline{MF} through G . Let the intersection of this perpendicular and \overline{MF} be called a point H . The segment \overline{HF} is the harmonic mean, which is $\frac{2ab}{a+b}$.

Justification. By Theorem 7.1, we can construct a line perpendicular to \overline{MF} through the point G .

Proof. Since \overline{GH} , with respect to \overline{FG} , is a perpendicular to \overline{MF} (with respect to \overline{MG}), $m\angle GHF = m\angle FGM = 90^\circ$. Since $m\angle FGH = m\angle FGM$ and $m\angle F = m\angle F$, $\triangle FGH$ is similar to $\triangle FGM$ because of Theorem 12.3 (AA Similarity Theorem). Therefore, $\frac{FH}{FG} = \frac{FG}{FM}$. As we know from 4.2, \overline{FG} is the geometric mean of a and b ; therefore, $FG = \sqrt{ab}$. Because MD is the arithmetic mean of a and b and $MD = \frac{a+b}{2}$, $FM = MD$ because they are the same radius of the same circle, so $FM = MD = \frac{a+b}{2}$. Substituting $FG = \sqrt{ab}$ and $FM = \frac{a+b}{2}$ into the equation $\frac{FH}{FG} = \frac{FG}{FM}$ yields:

$$\begin{aligned} FH &= \frac{FG^2}{FM} \\ &= \frac{ab}{\frac{a+b}{2}} \\ &= \frac{2ab}{a+b}. \end{aligned}$$

Thus, \overline{FH} is the harmonic mean. □

4.4 AM-GM-HM Inequality

Now that we have constructed the three means, notice that the AM-GM-HM inequality directly follows from Figure 5. Recall that the arithmetic mean is \overline{AM} , which is equal to \overline{MF} because they are both radii of the circle. Also recall that the geometric mean is \overline{GF} and the harmonic mean is \overline{GH} . $MF > GF$ because \overline{MF} is the hypotenuse of $\triangle MFG$ so it is strictly greater than either leg by Corollary 5.17. Similarly, $GF > GH$ because \overline{GF} is the hypotenuse of $\triangle FGH$.

5 Application

Construct a square with the same area as a given rectangle.

With our newfound knowledge of the geometric mean, interesting geometric problems that arise can be viewed through another angle besides mere application of algebra. Consider for example a rectangle with length a and width b so that its area is the product of the length and width, ab . As any student of geometry should know, a rectangle is surely not a square. But what if one needs to construct a square such that its area is equal to that of our rectangle? While one may quickly see that this problem can be solved by elementary algebra and knowledge of the fact that the area of a square with side length x is x^2 , the underlying principle is that the length of each side of our newly crafted square will be the geometric mean of a and b (the length and width, respectively) of our rectangle:

$$\begin{aligned} x^2 &= ab \\ x &= \sqrt{ab}. \end{aligned}$$

6 References

[1] Kalid Azad, “How to Analyze Data Using the Average,” <<http://betterexplained.com/articles/how-to-analyze-data-using-the-average/>>, 14 March 2008. 16 May 2013.

[2] Alexander Bogomolny, “Averages, Arithmetic and Harmonic Means from Interactive Mathematics Miscellany and Puzzles,” <<http://www.cut-the-knot.org/arithmetic/HarmonicMean.shtml>>. 23 May 2013.

[3] John Lee, *Axiomatic Geometry*, AMS (2013).

[4] Nathan Wisdom, “An Exploration of the Pythagorean Means,” <http://jwilson.coe.uga.edu/EMAT6680Fa08/Wisdom/EMAT6690/Investigate%20Means/Means_njw.html>. 13 May 2013.