Enumerative Geometry
and the Shapiro–Shapiro Conjecture

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Enumerative geometry

Some questions about 3D geometry:

1. How many lines meet four general lines?
2. How many lines lie on a smooth cubic surface?
3. How many rational cubic curves meet 12 lines?
Enumerative geometry

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(Credit: Greg Egan)
Enumerative geometry

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Prettier than 80160 twisted cubics.
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To answer these questions, we study moduli spaces:

- The space of lines $Gr(2, n)$
- The space of curves $\mathcal{M}_g$
- The space of maps of curves $\mathcal{M}_{0,n}(\mathbb{P}^3, 3)$
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Hilbert’s 15th Problem (1900): “To establish rigorously ... the enumerative calculus developed by [Schubert].”

Mumford (1983): “We take as a model for [enumeration on $M_g$] the enumerative geometry of the Grassmannians.”
Modern questions also go beyond enumeration

- Euler characteristic, genus, ... for positive-dimensional solution spaces

- Explicit topology of moduli spaces (e.g. as CW-complexes)

- Deforming solutions
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- Deforming solutions

**Goal of today**

Pass from **counting** to **describing topology**.
Part 1. Counting
Counting lines and planes

How many lines meet four general lines?

Schubert calculus says: 2 solutions – enumerated by the set \{1234, 1324\}.

Caveat: in general, over \mathbb{C} (in \mathbb{C}P^3, with multiplicity,...).
Counting lines and planes

How many lines meet four general lines?

Schubert calculus says: 2 solutions – enumerated by the set

\[
\left\{ \begin{array}{cc}
1 & 2 \\
3 & 4
\end{array} \right\}, \quad \left\{ \begin{array}{cc}
1 & 3 \\
2 & 4
\end{array} \right\}.
\]

Caveat: in general, over $\mathbb{C}$ (in $\mathbb{CP}^3$, with multiplicity, ...).
Schubert problems, \( k \)-planes, flags

The **Grassmannian** is the space of planes:

\[
Gr(k, n) = \{ \text{vector subspaces } S \subset \mathbb{C}^n : \dim(S) = k \}.
\]
Schubert problems, $k$-planes, flags

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Simplest (codimension 1) **“Schubert problem”** for planes:

$$X^\Box(F_{n-k}) = \{ S \in Gr(k, n) : S \cap F_{n-k} \neq 0 \}.$$ 

In $\mathbb{P}^3$: Lines meeting a given line.
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**Theorem.** For *general* choices of complementary planes $F^{(i)}$,

$$Z_{k,n} := X^\square(F^{(1)}_{n-k}) \cap \cdots \cap X^\square(F^{(k(n-k))}_{n-k}) \text{ is finite}$$
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and counted by **standard Young tableaux**:

\[ \#Z_{k,n} = \#\text{SYT}(\square) = \left\{ \begin{array}{c}
1 & 2 & 4 \\
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Choosing flags

**Theorem.** For *general* choices of $F^{(i)}$,

$$Z_{k,n} = X^\Box(F_{n-k}^{(1)}) \cap \cdots \cap X^\Box(F_{n-k}^{(k(n-k))})$$

is **finite** and counted by **standard Young tableaux**:

$$\#Z_{k,n} = \#\text{SYT}(\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array}) = \left\{ \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array}, \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}, \cdots \right\}.$$
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General Schubert problems: consider $S \cap \mathcal{F}$, for a **complete flag**:

$$\mathcal{F} : F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n.$$
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General Schubert problems: consider $S \cap \mathcal{F}$, for a complete flag:

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Problems:

1. In bad cases, $Z_{k,n}$ might have multiplicity (or be infinite).
2. No canonical bijection $Z_{k,n} \leftrightarrow \text{SYT}(\begin{array}{cccc}1 & 2 & 4 \\ 3 & 5 & 6 \end{array})$ in general.
Tangent flags to the rational normal curve
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- **The rational normal curve** in $\mathbb{P}^{n-1}$:
  
  $$
  \mathbb{P}^1 \hookrightarrow \mathbb{P}(\mathbb{C}^n) = \mathbb{P}^{n-1} \text{ by }
  t \mapsto [1 : t : t^2 : \cdots : t^{n-1}]
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- (Maximally) **tangent flag** $\mathcal{F}(t)$, $t \in \mathbb{P}^1$: 
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- Eisenbud–Harris '83:
  Solutions to these problems have the expected dimension.
Schubert calculus over $\mathbb{R}$?

Conjecture (Shapiro–Shapiro '95)

For any choice of distinct real $t_1, \ldots, t_N \in \mathbb{RP}^1$,

$$Z_{k,n} = X^\Box(\mathcal{F}(t_1)_{n-k}) \cap \cdots \cap X^\Box(\mathcal{F}(t_N)_{n-k})$$

consists entirely of real, multiplicity-free points.
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Case $k = 2$ established earlier:

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▶ The cardinality of $Z_{k,n}$ is always exactly $\# \text{SYT}$.

▶ This suggests there may be a canonical bijection $Z_{k,n} \leftrightarrow \text{SYT}$. 
Combinatorial consequences

Theorem (M–T–V ’05)

For any choice of distinct real \( t_1, \ldots, t_N \in \mathbb{RP}^1 \),

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- The cardinality of \( Z_{k,n} \) is always exactly \(#\text{SYT}\).
- This suggests there may be a canonical bijection

\[
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Part 2. Labeling
Families of geometry problems

We want to study $Z_{k,n}$ for every possible choice of $t_i$'s.

**Configuration space:** $P_N(\mathbb{R}) = \{\text{sets of distinct } t_1, \ldots, t_N \in \mathbb{R}\}$. 
Families of geometry problems

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$Z_{k,n}$

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Fiber is $Z_{k,n} = X_{\square}(\mathcal{F}(t_1)) \cap \cdots \cap X_{\square}(\mathcal{F}(t_N))$. 
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Configuration space: $P_N(\mathbb{R}) = \{\text{sets of distinct } t_1, \ldots, t_N \in \mathbb{R}\}$.

Fiber is $Z_{k,n} = \bigcap X^\square(\mathcal{F}(t_1)) \cap \cdots \cap X^\square(\mathcal{F}(t_N))$.

Observation: $P_N(\mathbb{R})$ is contractible. No monodromy!

If we can label one fiber by tableaux, we can label all of them.
How to find combinatorics in geometry

**Key idea**

Degenerate the problem until it breaks into pieces.
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Take \((t_1, \ldots, t_N) = (z, z^2, \ldots, z^N)\) and take \(\lim_{z \to 0}\)

What will happen to \(Z_k, n\) at \(z = 0\)?
How to find combinatorics in geometry

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What will happen to \(Z_{k,n}\) at \(z = 0\)?
Lines through 4 given lines, redux

Note: \( \{ \text{lines in } \mathbb{P}^3 \} = Gr(2, 4) \approx \left\{ \begin{bmatrix} 0 & 1 & * & * \\ 1 & 0 & * & * \end{bmatrix} \right\} \).
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Set up \( Z_{2,4} \) using \textbf{tangent lines} from the flags \( \mathcal{F}(t) \):

\[
Z_{2,4} = X^\square(\mathcal{F}(z)) \cap \cdots \cap X^\square(\mathcal{F}(z^4)) \subset Gr(2, 4)
= \{ \text{lines in } \mathbb{P}^3 \text{ meeting 4 given (tangent) lines} \}\.
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Set up \( Z_{2,4} \) using tangent lines from the flags \( F(t) \):

\[
Z_{2,4} = X^\Box(F(z)) \cap \cdots \cap X^\Box(F(z^4)) \subset \text{Gr}(2, 4)
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\[
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\]

\[
\text{Pr}(\mathbb{R}) \quad Z_{2,4} \subset \text{Gr}(2,4) \quad \text{(two solutions)}
\]
Tableau labels from Plücker coordinates

Limiting matrix:

\[
\begin{bmatrix}
0 & 1 & \approx z^1 & \approx z^3 \\
1 & 0 & \approx z^4 & \approx z^6 \\
\end{bmatrix}
\]

Plücker coordinates (minors) on \(Gr(2,4)\):

\[
\begin{align*}
det_{14} &= O(z^3) \\
det_{12} &= 1 \\
det_{13} &= O(z^1) \\
det_{23} &= O(z^4) \\
det_{24} &= O(z^6) \\
det_{34} &= O(z^{10})
\end{align*}
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Plücker coordinates (minors) on \( Gr(2, 4) \):

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\begin{align*}
det_{12} &= z^0 \\
det_{13} &= O(z^1) \\
det_{14} &= O(z^{1+2}) \\
det_{23} &= O(z^{1+3}) \\
det_{24} &= O(z^{1+2+3}) \\
det_{34} &= O(z^{1+2+3+4})
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Combinatorics and geometry

**Theorem.** This procedure gives a bijection \( Z_{k,n} \leftrightarrow \text{SYT}(\square) \).
(Purbhoo ’09, Speyer ’14)
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And: moving $\{t_i\}$ on $\mathbb{RP}^1$ changes the labels by known algorithms!

![Diagram of RP^1 with labels for tableau promotion and tableau evacuation]
Combinatorics and geometry

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And: moving $\{t_i\}$ on $\mathbb{RP}^1$ changes the labels by known algorithms!

And more:

- Topology and genus when $\dim(Z) = 1$ (Levinson, Gillespie–L)
- Orthogonal Grassmannians (Purbhoo, Gillespie–L–Purbhoo)
- Vector bundles on $\overline{M}_{0,n}$ (Kamnitzer, Rybnikov)
Part 3. Topology!
A challenge and a new approach

Theorem (M–T–V ’05, ’09)

For \( t_1, \ldots, t_N \in \mathbb{RP}^1 \), \( Z_{k,n} \) consists of real, multiplicity-free points.

Challenge for geometers:

- M–T–V proof uses integrable systems, the Bethe ansatz
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- Subsequent geometry work used M–T–V as black box.
- Many open generalizations of interest!
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It turns out there is a topological / geometric approach.

\[
\begin{pmatrix}
1 & 2 & 5 \\
3 & 4 & 6
\end{pmatrix}, \quad
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\end{pmatrix}, \ldots
\]

Oriented Young tableaux.
Generalization: complex conjugate points on $\mathbb{P}^1$

Before: Defined $Z_{k,n}$ using **real** points $t_i \in \mathbb{R}\mathbb{P}^1$. 
Generalization: complex conjugate points on $\mathbb{P}^1$

Before: Defined $Z_{k,n}$ using real points $t_i \in \mathbb{RP}^1$.

Now: $Z_{k,n} = \bigcap_{i=1}^{n_1} X^\square(t_i) \cap \bigcap_{j=1}^{n_2} X^\square(t_j) \cap X^\square(t_j)$.

- $\bigcap_{i=1}^{n_1}$ real
- $\bigcap_{j=1}^{n_2}$ complex conjugate pairs

Mixed configuration space: For a partition $\mu = (2n_2, 1n_1)$, let $\mathcal{P}(\mu) = \{\text{sets of } n_1 \text{ distinct points on } \mathbb{R}, \ n_2 \text{ complex conjugate pairs on } \mathbb{C} \setminus \mathbb{R}\} \subseteq \mathcal{P}_r(\mathbb{C}) \ (r = 2n_2 + n_1)$.

(Base case: $\mu = (1N)$, all real $t_i$.)

Generalization: complex conjugate points on \( \mathbb{P}^1 \)

Before: Defined \( Z_{k,n} \) using \textbf{real} points \( t_i \in \mathbb{RP}^1 \).

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\textbf{Mixed configuration space:}
For a partition \( \mu = (2^{n_2}, 1^{n_1}) \), let

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\left. \text{and } n_2 \text{ complex conjugate pairs on } \mathbb{C} \setminus \mathbb{R} \right\}
\]

\[
\subseteq P_r(\mathbb{C}) \ (r = 2n_2 + n_1).
\]

(Base case: \( \mu = (1^N), \text{ all real } t_i. \))
Topological and algebraic degrees

How many real points in $Z_{k,n}$ for $(t_1, \ldots, t_N) \in P(\mu)$?

- Upper bound from (algebraic) degree $= \#SYT(\mu)$. 

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- We use a careful twist of standard orientation on $Gr(k,n)$. 

Topological and algebraic degrees

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- Lower bound from topological degree... (=?):

![Diagram showing real points in $Z_{k,n}$ and $P(\mu)$ with algebraic and topological degrees indicated.]

Algebraic degree: 3

Topological degree: 1

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\[ Z_{k,n} \]

![Diagram](image)

Algebraic degree: 3

\[ P(\mu) \]
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![Diagram]

\( Z_{k,n} \)

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### The topological degree of $\mathcal{Z}_{k,n}$

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<tr>
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</thead>
<tbody>
<tr>
<td>$\square$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\square$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>$\square$</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\square$</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$\square$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

For $S_{k(n-k)}$, let $\square$ be the $k \times (n - k)$ rectangle, $\mu = (2^{n_2}, 1^{n_1})$. 

Character table of $S_4$. $(\chi^\lambda(\mu))$
The topological degree of $\mathcal{Z}_{k,n}$

Character table of $S_4$. $(\chi^\lambda(\mu))$

<table>
<thead>
<tr>
<th>$\lambda, \mu$</th>
<th>(4)</th>
<th>(3, 1)</th>
<th>(2$^2$)</th>
<th>(2, 1$^2$)</th>
<th>(1$^4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{array}{c} \text{□□□□} \ \text{□□□} \ \text{□□□} \ \text{□□□} \end{array}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\begin{array}{c} \text{□□□□} \ \text{□□□} \ \text{□□□} \ \text{□□□} \end{array}$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>$\begin{array}{c} \text{□□□□} \ \text{□□□} \ \text{□□□} \ \text{□□□} \end{array}$</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\begin{array}{c} \text{□□□□} \ \text{□□□} \ \text{□□□} \ \text{□□□} \end{array}$</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$\begin{array}{c} \text{□□□□} \ \text{□□□} \ \text{□□□} \ \text{□□□} \end{array}$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

For $S_{k(n-k)}$, let $\begin{array}{c} \text{□□□□} \\ \text{□□□} \\ \text{□□□} \\ \text{□□□} \end{array}$ be the $k \times (n - k)$ rectangle, $\mu = (2^{n_2}, 1^{n_1})$.

**Theorem (L, Purbhoo ‘19)**

There is an orientation, the **character orientation**, such that the family $\mathcal{Z}_{k,n}$ has topological degree $\chi^{\begin{array}{c} \text{□□□□} \\ \text{□□□} \\ \text{□□□} \\ \text{□□□} \end{array}}(\mu)$ over $P(\mu)$. 
### Signed Young tableaux

<table>
<thead>
<tr>
<th>Theorem (L, Purbhoo ‘19)</th>
</tr>
</thead>
</table>

*There is an orientation, the **character orientation**, such that the family $Z_{k,n}$ has topological degree $\chi^{\mu}(\mu)$ over $P(\mu)$.***

**Murnaghan–Nakayama rule for $\chi^{\lambda}(\mu)$, $\mu = (2^n, 1^n)$:**

\[
\chi^{\lambda}(\mu) = \sum_{T} (-1)^{\#(T)}: \mu\text{-domino tableaux (}+\text{)}
\]

**Special case:** $\mu = (1^N)$, no dominos

$\Rightarrow \chi^{1^N} = \#\text{SYT}.$

**Corollary:** Shapiro–Shapiro Conjecture.
Signed Young tableaux

Theorem (L, Purbhoo '19)

There is an orientation, the character orientation, such that the family $\mathcal{Z}_{k,n}$ has topological degree $\chi_{\lambda}(\mu)$ over $P(\mu)$.

Murnaghan–Nakayama rule for $\chi_{\lambda}(\mu)$, $\mu = (2^{n_2}, 1^{n_1})$:

$$
\chi_{\lambda}(\mu) = \sum_T (-1)^{#(T)} \mu\text{-domino tableaux } (+) \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 6
\end{array}, (-) \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & 6
\end{array}, \ldots
$$

shape($T$) = $\lambda$.
Signed Young tableaux

**Theorem (L, Purbhoo ‘19)**

There is an orientation, the **character orientation**, such that the family $Z_{k,n}$ has topological degree $\chi(\mu)$ over $P(\mu)$.

Murnaghan–Nakayama rule for $\chi^\lambda(\mu)$, $\mu = (2^{n_2}, 1^{n_1})$:

$$\chi^\lambda(\mu) = \sum_T (-1)^{\#(T)}: \mu\text{-domino tableaux } (+) \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array}, (-) \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & 6 \end{array}, \cdots$$

where $\text{shape}(T) = \lambda$.

- **Special case**: $\mu = (1^N)$, no dominos $\rightsquigarrow \chi(1^N) = \#\text{SYT}$.
- **Corollary**: Shapiro–Shapiro Conjecture.
Labeling \( Z_{k,n} \) by signed Young tableaux

**Proof sketch:**

- Label limit fibers by tableaux.
- Track +/- signs along a network of paths.
Labeling $Z_{k,n}$ by signed Young tableaux

Proof sketch:

- Label limit fibers by tableaux.
- Track $+/−$ signs along a network of paths.

Case 1: $\frac{1}{2} \leftrightarrow \frac{1}{2}$ / $12 \leftrightarrow 12$

$P(\mu) \quad P(\mu = (1^6)) \quad P(\mu' = (2, 1^4))$
Labeling $Z_{k,n}$ by signed Young tableaux

**Proof sketch:**

- Label limit fibers by tableaux.
- Track $+/-$ signs along a network of paths.

**Case 2:** $\begin{array}{c} 2 \ 3 \\ 1 \ 2 \ 4 \ 5 \ 6 \end{array} \leftrightarrow \begin{array}{c} 3 \\ 1 \ 2 \ 4 \ 5 \ 6 \end{array}$

$P(\mu) \leftarrow \begin{array}{c} 2 \ 3 \\ 1 \ 2 \ 4 \ 5 \ 6 \end{array} \Rightarrow (\varnothing)$

$P(\mu) \leftarrow \begin{array}{c} 2 \ 3 \\ 1 \ 2 \ 4 \ 5 \ 6 \end{array} \Rightarrow (\varnothing)$

$P(\mu = (1^6)) \leftarrow (\varnothing)$

$P(\mu = (2, 1^4)) \leftarrow (\varnothing)$

$P(\mu') \leftarrow (\varnothing)$
Open questions

- (Representation theory).
  Do all $S_N$ characters $\chi^\lambda(\mu)$ give topological degrees of real Schubert problems?

- (Complex geometry).
  Explicit geometry over $P(\mu)$ for $\mu \neq (1^N)$?

- (Stable curves).
  How does the geometry look over the moduli space $\overline{M}_{0,N}$?
  - $\overline{M}_{0,N}(\mathbb{R})$ is non-orientable!

Many interesting relationships to find between geometry and combinatorics.
Thank you!