John M. Lee’s *Introduction to Riemannian Manifolds (2nd ed)*

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Notations and Conventions

**Notation:** For any integer \( n \geq 1 \), \( \mathbb{H}^n \) denotes the upper-half space of \( \mathbb{R}^n \):

\[
\mathbb{H}^n = \{ x \in \mathbb{R}^n : x^n \geq 0 \}
\]

**Notation:** The notation \( \mathbb{R}_+ \) denotes the set of positive real numbers:

\[
\mathbb{R}_+ = \{ x \in \mathbb{R} : x > 0 \}.
\]

**Notation:** The notation \( \mathbb{Z}_+ \) denotes the set of positive integers:

\[
\mathbb{Z}_+ = \{1,2,3,...\}.
\]

**Convention:** A neighborhood in a topology always mean an *open* neighborhood.

**Convention:** All appropriate structures are smooth (\( \mathcal{C}^\infty \)) unless stated otherwise. I’ll often still include the word *smooth* for emphasis.

**Definition:** A regular domain is a smooth properly embedded submanifold with boundary of codimension zero.

Chapter 6

Boundary Normal Coordinates (Example 6.44 [Page 183])

Here I work out the details of the construction of boundary normal coordinates (boundary coordinates that are also semi-geodesic coordinates). Recall that all structures here are considered smooth unless stated otherwise. First let’s prove a fact about extending Riemannian metrics.

**Lemma 1:** Suppose that \( g \) is a Riemannian metric on an open subset \( U \) of \( \mathbb{H}^n \). Then there exists a neighborhood \( V \) of \( U \) open in \( \mathbb{R}^n \) and a Riemannian metric \( \tilde{g} \) on \( V \) such that \( \tilde{g} \) extends \( g \) (i.e. the inclusion \( i : U \to V \) is an isometry).
**Proof:** Intuitively speaking, as a first step let’s locally extend $g$ to a neighborhood of any point $p \in U$ that’s open $\mathbb{R}^n$. Precisely, take any point $p \in U$ and let’s define a Riemannian metric $\bar{g}_p$ on a neighborhood $W_p$ of $p$ that’s open in $\mathbb{R}^n$ as follows:

*Case $p \in U^{\text{int}}$ in $\mathbb{R}^n$’s topology:* In this case let $W_p \subseteq U^{\text{int}}$ be a neighborhood of $p$ and set $\bar{g}_p = g$ on $W_p$.

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*Case $p \in \partial U$ in $\mathbb{R}^n$’s topology:* Let’s first write $g$ in its Euclidean components:

$$g = g_{ij} \, dx^i \otimes dx^j.$$

Now, take any index $(i,j)$ such that $i \leq j$. Since the component $g_{ij}$ is smooth, there exists a smooth function $\bar{g}_{ij} : W_{ij} \to \mathbb{R}$ on a neighborhood $W_{ij}$ of $p$ open in $\mathbb{R}^n$ that agrees with $g_{ij}$ on $W_{ij} \cap U$. Find such a $\bar{g}_{ij}$ for all indices $(i,j)$ such that $i \leq j$. For the other indices $(i,j)$ where $i > j$, simply set $g_{ij} : W_{ij} \to \mathbb{R}$ to be equal to $g_{ji} : W_{ji} \to \mathbb{R}$. Then, if we define

$$W = \bigcap_{i,j \leq n} W_{ij},$$

we get a smooth covariant symmetric 2-tensor field $h$ on $W$ given by:

$$h = \bar{g}_{ij} \, dx^i \otimes dx^j$$

that agrees with $g$ on $W \cap U$. Now, $h$ is positive definite at $p$ since it’s equal to $g$ at that point. So by Sylvester’s criterion and the continuity of the determinants of the principal minors of the matrix $[\bar{g}_{ij}]$ we see that we can furthermore shrink $W$ so that $h$ is positive definite on $W$. After shrinking $W$ in this manner, set $W_p = W$ and $\bar{g}_p = h$.

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Great! Notice that in both cases $\bar{g}_p$ are local extensions of $g$ in the sense that the inclusion map $i : W_p \cap U \to W_p$ is an isometry. Now let’s use these local extensions to construct the $V$ and $\bar{g}$ desired in the lemma. Let $V \subseteq \mathbb{R}^n$ be the following open neighborhood of $U$ in $\mathbb{R}^n$:

$$V = \bigcup_{p \in U} W_p.$$

Let $\{\psi_p : V \to \mathbb{R}\}_{p \in U}$ be a smooth partition of unity over $V$ subordinate to the open cover $\{W_p : p \in U\}$. Set $\bar{g}$ to be the following Riemannian metric over $V$:

$$\bar{g} = \sum_{p \in U} \psi_p \cdot \bar{g}_p.$$
This is of course well defined since the $\psi_p$’s are locally finite. And since each $\tilde{g}_p$ locally extends $g$ and the $\psi_p$’s add up to one, we get that $\tilde{g}$ extends $g$ in the sense described in the lemma. Thus this is the $V$ and $\tilde{g}$ that we wanted.

Now let’s prove a version of the above lemma on manifolds.

**Lemma 2:** Suppose that $\tilde{M}$ is a smooth manifold (without boundary) and that $M \subseteq \tilde{M}$ is a regular domain in $\tilde{M}$. Suppose also that $M$ is endowed with a Riemannian metric $g$. Then there exists a Riemannian metric $\tilde{g}$ on $\tilde{M}$ that extends $g$ (i.e. the inclusion $i : M \to \tilde{M}$ is an isometry).

**Proof:** This is proved similarly to the previous lemma. Pick any point $p \in \tilde{M}$. There are two cases that can happen here: $p \in M$ or $p \in M^c$. Suppose that the first case happens: $p \in M$. Then since $M$ is an embedded submanifold with boundary in $\tilde{M}$, there exists a chart $(W_p, \varphi_p)$ of $\tilde{M}$ such that either $\varphi_p$ is an interior chart of $M$ as well or $W_p \cap M$ is the half-slice $x^n \geq 0$ where $(x^i)$ are the components of $\varphi_p$. Construct a Riemannian metric $\tilde{g}_p$ over $W_p$ as follows:

*Case $\varphi_p$ is an interior chart of $M$:* Simply set $\tilde{g}_p = g$ over $W_p$.

*Case $W_p \cap M$ is the half-slice $x^n \geq 0$:* Let $\tilde{U}_p = \text{range } \varphi_p$. Then we have that the restriction $\varphi_p : W_p \cap M \to \tilde{U}_p \cap \mathbb{R}^n$ is a smooth chart of $M$. Since $g$ is smooth over $M$, we have that $\tilde{g} = \varphi_p^{-1} \cdot g$ is smooth over $\tilde{U}_p \cap \mathbb{R}^n$. By the previous lemma we know that there exists a smooth extension $\tilde{g}_E$ of $\tilde{g}$ onto a neighborhood $\tilde{\mathcal{V}} \subseteq \tilde{U}_p$ of $\tilde{U}_p \cap \mathbb{R}^n$ open in $\mathbb{R}^n$ (the condition $\tilde{\mathcal{V}} \subseteq \tilde{U}_p$ is a trivial modification to the previous lemma). Redefine $W_p$ to instead be $\varphi^{-1}[\tilde{\mathcal{V}}]$ and set $\tilde{g}_p = \varphi^* \tilde{g}_E$ on $W_p$.

Notice that in both cases $\tilde{g}_p$ are local extensions of $g$ in the sense that the inclusion map $i : W_p \cap M \to W_p$ is an isometry.

Now suppose that the other case happens: $p \in M^c$. Since $M$ is closed in $\tilde{M}$ (since it’s properly embedded) there exists a neighborhood $W_p$ of $p$ open in $\tilde{M}$ that is disjoint from $M$. Let $W_p$ be any such neighborhood and let $\tilde{g}_p$ be any Riemannian metric over $W_p$.

Great! Let $\{\psi_p : \tilde{M} \to \mathbb{R}\}_{p \in \tilde{M}}$ be a smooth partition of unity over $\tilde{M}$ subordinate to the open cover $\{W_p : p \in \tilde{M}\}$ of $\tilde{M}$. Finally, let $\tilde{g}$ denote the following Riemannian metric over $\tilde{M}$:

$$\tilde{g} = \sum_{p \in \tilde{M}} \psi_p \cdot \tilde{g}_p.$$
Notice that by construction, for any $q \in M$ and any $p \in M^c$, $\psi_p : \tilde{g}_p$ is equal to zero at $q$ since $\text{supp} \psi_p \subseteq W_p$ is disjoint from $M$. This combined with the facts that the $\tilde{g}_p$ for $p \in M$ locally extend $g$ and that the $\psi_p$'s add up to one shows that $\tilde{g}$ extends $g$ in the sense described in the lemma. Thus this is the $\tilde{g}$ that we wanted.

Now let's discuss the main topic of this section: the construction of boundary local coordinates. Suppose that $(M, g)$ is a Riemannian manifold. Take any point $p \in \partial M$ where we want to construct boundary normal coordinates. Let $D(M)$ denote the double of $M$ and let $e : M \to D(M)$ be a proper smooth embedding. Then we have that the image $\tilde{M} = e[M]$ is a regular domain in $D(M)$. Letting $\tilde{e} : M \to \tilde{M}$ denote the diffeomorphism obtained by shrinking the image of $e$ to its range, we have that $\tilde{M}$ naturally inherits the Riemannian metric $\tilde{g} = \tilde{e}^{-1} \ast g$ from $M$. Observe also that $\partial \tilde{M} = e[\partial M]$ since $e = \tilde{e}$ over $M$ and diffeomorphisms take boundary points to boundary points. In particular we have that $\tilde{p} = e(p)$ is in $\partial \tilde{M}$.

Now, by the previous lemma we know that there exists a Riemannian metric $g_{D(M)}$ on $D(M)$ that extends $\tilde{g}$. Let $U$ be a normal neighborhood of the embedded hypersurface $\partial \tilde{M}$ in $D(M)$. Let $N : W \to TD(M)$ be a smooth vector field over a neighborhood $W \subseteq \partial \tilde{M}$ of $\tilde{p}$ open in $\partial \tilde{M}$ that is inward pointing with respect to $\tilde{M}$ (it's not hard to see that such an $N$ exists: look in half-slice coordinates for instance). By shrinking if necessary, let's furthermore assume that $W$ is the domain of a smooth chart $(\bar{W}, \psi)$ of $\partial \tilde{M}$. Letting $E : \mathcal{E}_{\partial \tilde{M}} \to U$ denote the normal exponential map of $\partial \tilde{M}$ that maps diffeomorphically onto $U$ we have that $\psi$ and $N$ together generate a Fermi chart $(\mathcal{O}, \phi)$ of $D(M)$ by the equation:

$$\phi^{-1}(x^1, ..., x^{n-1}, x^n) = E \left(x^n N_{\psi^{-1}(x^1, ..., x^{n-1})}\right).$$

Great! With $(\mathcal{O}, \phi)$ in hand we are ready to construct the boundary normal coordinates for $M$ in a neighborhood of $p$. First let's observe one thing. Let $\tilde{U} \subseteq \mathbb{R}^n$ denote the range of $\phi$. Then:

**Claim:** $\phi$ maps $\mathcal{O} \cap \tilde{M}$ to $\tilde{U} \cap \{x^n \geq 0\}$ and $\mathcal{O} \cap \tilde{M}^c$ to $\tilde{U} \cap \{x^n < 0\}$.

**Proof of Claim:** Fix any point of the form $(x^1, ..., x^{n-1}, 0) \in \tilde{U}$. Recall that by definition, the intersection of normal neighborhoods of embedded submanifolds with fibers of the normal bundle is starshaped with respect to zero. In this case this implies that the set of $t \in \mathbb{R}$ such that $(x^1, ..., x^{n-1}, t) \in \tilde{U}$ is some interval $(a, b)$ that contains zero. Consider then the smooth curve $\gamma : (a, b) \to D(M)$ given by $t \mapsto \phi^{-1}(x^1, ..., x^{n-1}, t)$. The claim will be proven if we show that $\gamma(t) \in \tilde{M}$ for $t \geq 0$ and $\gamma(t) \in \tilde{M}^c$ if $t < 0$. Let's first prove that $\gamma(t) \in \tilde{M}$ for $t \geq 0$. By definition of the normal exponential map, we have that $\gamma(0) \in \partial \tilde{M} \subseteq \tilde{M}$. Furthermore, by the injectivity of the exponential map we have that $\gamma(t) \notin \partial \tilde{M}$ for $t > 0$. So for $t > 0$, or in other words $t \in (0, b)$, $\gamma(t)$ lies in the two disjoint open set $\tilde{M}^\text{int}$ and $\tilde{M}^c$ in $D(M)$. Since the interval $(0, b)$ is connected we must have that $\gamma(t)$ lies exclusively in either $\tilde{M}^\text{int}$ and $\tilde{M}^c$ on this interval. But notice that since
\[
\gamma'(0) = N_{\phi^{-1}(x^1, \ldots, x^{n-1}, 0)}
\]

and by construction this vector is inward pointing with respect to \(\tilde{M}\), we see that \(\gamma(t)\) is guaranteed to lie in \(\tilde{M}\) on a sufficiently small interval of the form \([0, \varepsilon)\) where \(\varepsilon > 0\). So we must indeed have that \(\gamma(t) \in \tilde{M}\) for \(t \geq 0\).

The fact that \(\gamma(t) \in \tilde{M}^c\) for \(t < 0\) is proved similarly. Indeed: by the injectivity of the exponential map we have that \(\gamma(t) \notin \partial \tilde{M}\) for \(t < 0\). Since \(\tilde{M}^\text{int}\) and \(\tilde{M}^c\) are disjoint open set in \(D(M)\) and \((a, 0)\) is connected, we have that \(\gamma(t)\) must lie in either \(\tilde{M}^\text{int}\) or \(\tilde{M}^c\) for \(t < 0\). But the above displayed equation implies that \(\gamma(t)\) is guaranteed to lie in \(\tilde{M}^c\) on a sufficiently small interval of the form \((-\delta, 0)\) where \(\delta > 0\). So indeed \(\gamma(t) \in \tilde{M}^c\) for \(t < 0\). This proves the claim.

Thus \((O, \phi)\) is a half-slice chart of \(\tilde{M}\) in \(D(M)\). So letting \(\tilde{O} = O \cap \tilde{M}\) and \(\tilde{\phi} = \phi|_{\tilde{O} \cap \tilde{M}}\) be the restriction of this chart to \(O \cap \tilde{M}\), we get a chart \((\tilde{O}, \tilde{\phi})\) of \(\tilde{M}\). Composing this with \(\tilde{e}\) over \(W = \tilde{e}^{-1}[\tilde{O}]\):

\[
\Psi = \tilde{\phi} \circ \tilde{e}|_W
\]

then gives a chart \((W, \Psi)\) of \(M\) in a neighborhood of \(p\).

Ok, we’re done with the construction of this chart and so now you might ask: what’s so special about it? It’s special because its coordinates are semi-geodesic coordinates. In other words, it gives us the boundary normal coordinates that we wanted to construct. Let’s see why. First let me make the technical comment that since Jack Lee only discusses geodesics on manifolds without boundary at this point, whenever I say that boundary coordinates are semi-geodesic coordinates, I mean that their restriction to the interior of the manifold are semi-geodesic coordinates. Ok, back to proving my claim. Notice that the curves \(t \mapsto \Psi^{-1}(x^1, \ldots, x^{n-1}, t)\) are geodesics for \(t > 0\) since they’re the images of the geodesics \(t \mapsto \phi^{-1}(x^1, \ldots, x^{n-1}, t)\) under the isometry \(\tilde{e}^{-1}\) (here I technically use the fact that the curves \(t \mapsto \phi^{-1}(x^1, \ldots, x^{n-1}, t)\) for \(t > 0\) lie in \(\tilde{M}^\text{int}\) which was shown in the proof of the above claim). Now, by Gauss’ Lemma for submanifolds we have that the curves \(t \mapsto \phi^{-1}(x^1, \ldots, x^{n-1}, t)\) are constantly perpendicular to the level sets \(x^n = t\) for \(t > 0\). Since \(\tilde{e}^{-1}\) is an isometry, we then get that the curves \(t \mapsto \Psi^{-1}(x^1, \ldots, x^{n-1}, t)\) are also perpendicular to the level sets \(x^n = t\) for \(t > 0\). Thus we indeed get that the coordinates of \(\Psi\) are semi-geodesic coordinates.