G. Friedlander and M. Joshi *Introduction to the Theory of Distributions* Notes

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Notations and Conventions

**Notation:** For any point $x \in \mathbb{R}^n$, we will denote its Euclidean length by $|x|$: Maintenance planned

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$  

**Notation:** For any $x \in \mathbb{R}^n$ and any $r > 0$, we let $B_r(x)$ denote the open ball of radius $r$ centered at $x$ with respect to the Euclidean distance:

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$  

**Notation:** Suppose that $X \subseteq Y \subseteq \mathbb{R}^n$ are open sets. For any function $\phi \in C_c^\infty(X)$, let $\phi^Y \in C_c^\infty(Y)$ denote the smooth extension of $\phi$ to $Y$ obtained by setting $\phi \equiv 0$ on $Y \setminus X$. It’s trivial to see then that $\text{supp } \phi = \text{supp } \phi^Y$ and hence $\phi^Y$ is indeed compactly supported as well.

**Notation:** We let $\mathbb{Z}_+$ stand for the positive integers: $\mathbb{Z}_+ = \{1,2,3,...\}$.

**Notation:** For any $n \in \mathbb{Z}_+$, let $I(n)$ denote the set of multi-indices of length $n$:

$$I(n) = \{(\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n : \text{each } \alpha_k \geq 0\}.$$
**Notation:** Let $\alpha, \beta \in I(n)$. Then

1.) $\alpha \leq \beta$ (resp. $\alpha < \beta$) means that each $\alpha_k \leq \beta_k$ (resp. $\alpha_k < \beta_k$).
2.) $\alpha!$ denotes $\alpha_1! \cdot \ldots \cdot \alpha_n!$.
3.) $|\alpha|$ denotes $\alpha_1 + \ldots + \alpha_n$.
4.) For any $x \in \mathbb{R}^n$, $x^\alpha$ denotes $x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$.
5.) For any sufficiently differentiable function or distribution $f$, $\partial^\alpha f$ denotes $\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f$.

**Notation:** We let the following denote the following spaces of *complex-valued* functions:

1.) $C^m(\mathbb{R}^n)$ denotes the space of $k$-times continuously differentiable functions over $\mathbb{R}^n$. In particular, $C^\infty(\mathbb{R}^n)$ denotes the space of smooth functions.
2.) $C^m_c(\mathbb{R}^n)$ denotes the space of $k$-times continuously differentiable functions over $\mathbb{R}^n$ with compact support. Sometimes $C^\infty_c(\mathbb{R}^n)$ is also denoted by $\mathcal{D}(\mathbb{R}^n)$.
3.) We let $\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing functions:

$$\mathcal{S}(\mathbb{R}^n) = \{ \phi \in C^\infty(\mathbb{R}^n) : |x^\alpha \partial^\beta \phi(x)| \to 0 \text{ as } |x| \to \infty \ \forall \alpha, \beta \in I(n) \}.$$ 

This space is called the **Schwartz space**.

**Notation:** We let the following denote the following space of distributions:

1.) $\mathcal{D}'(\mathbb{R}^n)$ denotes the space of distributions over $\mathbb{R}^n$.
2.) $\mathcal{E}'(\mathbb{R}^n)$ denotes the space of distributions over $\mathbb{R}^n$ with compact support.
3.) $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions over $\mathbb{R}^n$.

**Definition:** A function $f \in \mathbb{R}^n \to \mathbb{C}$ is said to be of **polynomial growth** if there exist $C, M \geq 0$ such that

$$|f(x)| \leq C (1 + |x|)^M \quad \forall x \in \mathbb{R}^n.$$

**Chapter 1**

**Principal Value Distribution $x^{-1}$ (Problem 1.3) (9/25/2020)**

The principle value distribution $x^{-1}$ in $\mathcal{D}'(\mathbb{R})$ is defined by:

$$\langle x^{-1}, \phi \rangle = \text{p.v.} \int \frac{\phi(x)}{x} \, dx = \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x} \, dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x} \, dx \right).$$

To see that this is indeed a distribution, take any compact subset $K \subseteq \mathbb{R}$. Let $b > 0$ be such that $K \subseteq [-b, b]$. For any $\phi \in C^\infty_c(K)$, we have that:

$$\lim_{\varepsilon \to 0^+} \left( \int_{-\varepsilon}^{-\infty} \frac{\phi(x)}{x} \, dx + \int_{\varepsilon}^{b} \frac{\phi(x)}{x} \, dx \right) = \lim_{\varepsilon \to 0^+} \left( \int_{-b}^{-\varepsilon} \frac{\phi(x)}{x} \, dx + \int_{\varepsilon}^{b} \frac{\phi(x)}{x} \, dx \right).$$
\[
= \lim_{\varepsilon \to 0^+} \left( \int_{\varepsilon}^{b} \frac{\phi(x) - \phi(-x)}{x} \, dx \right).
\]

Using the fact that:
\[
\phi(x) = \phi(0) + \int_{0}^{x} \phi'(s) \, ds = \phi(0) + x \int_{0}^{1} \phi'(xt) \, dt
\]
(the rightmost expression is in fact just the 1st order Taylor expression for \(\phi\) based at 0), we can rewrite the previous expression as:
\[
= \lim_{\varepsilon \to 0^+} \left( \int_{\varepsilon}^{b} \frac{\phi(0) + x \int_{0}^{1} \phi'(xt) \, dt - (\phi(0) - x \int_{0}^{1} \phi'(-xt) \, dt)}{x} \, dx \right)
\]
\[
\lim_{\varepsilon \to 0^+} \left( \int_{\varepsilon}^{b} \frac{x \int_{0}^{1} (\phi'(xt) + \phi'(-xt)) \, dt}{x} \, dx \right) = \lim_{\varepsilon \to 0^+} \left( \int_{\varepsilon}^{b} \int_{0}^{1} (\phi'(xt) + \phi'(-xt)) \, dt \, dx \right)
\]
\[
= \int_{0}^{b} \int_{0}^{1} (\phi'(xt) + \phi'(-xt)) \, dt \, dx.
\]

Where in the last equality I’ve used the Dominated Convergence Theorem with the observation that the integrand of the outside integral is bounded by \(2 \sup \phi' < \infty\) and that its integration domain is bounded (it’s overkill to cite DCT here though). Thus, for any \(\phi \in C_c^\infty(\mathbb{R})\) we have the estimate:
\[
|\langle x^{-1}, \phi \rangle| \leq 2b \sup \phi'
\]
and so \(x^{-1}\) is indeed a distribution.

Now, what is this distribution’s order? It turns out to be one. To prove this, first let’s observe that because of the above inequality we know that its order is less than or equal to one. So if we prove that it’s not equal to zero, then we’ll be done. To do this, for any positive numbers \(a, b, c \in \mathbb{R}: a, b, c > 0\) let \(\psi_{a, b, c} : \mathbb{R} \to \mathbb{R}\) be a compactly supported smooth bump function that satisfies:

1. \(\text{supp} \psi \subseteq [0, \infty)\),
2. \(\psi \equiv c\) on the interval \([a, b]\),
3. \(0 \leq \psi \leq c\) everywhere.

Jack Lee’s Smooth Manifolds book shows how to construct such a smooth bump function. A typical example looks like:
Notice that with these functions:

\[
\langle x^{-1}, \psi_{a,b,c} \rangle = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \frac{\psi_{a,b,c}(x)}{x} \, dx \geq \int_{a}^{b} \frac{c}{x} \, dx = c \ln \left( \frac{b}{a} \right).
\]

Now, consider the sequence of functions \( \{\psi_{a_k,b_k,c_k}\}_{k=1}^{\infty} \) with \( a_k = 1/k, b_k = 1, \) and \( c_k = e^{-k^2}. \) Then:

\[
\langle x^{-1}, \psi_{a_k,b_k,c_k} \rangle = \frac{1}{k} \ln \left( \frac{1}{e^{-k^2}} \right) = k \to \infty \quad \text{as} \quad k \to \infty.
\]

But if \( x^{-1} \) was an order 0 distribution, then the distribution “seminorm estimate” would tell us that \( \langle x^{-1}, \psi_{a_k,b_k,c_k} \rangle \to 0 \) as \( k \to \infty \) since:

\[
\sup \left| \psi_{a_k,b_k,c_k} \right| = \frac{1}{k} \to 0 \quad \text{as} \quad k \to \infty.
\]

So \( x^{-1} \) must indeed be an order 1 distribution.

A Distribution \( u \in \mathcal{D}'(\mathbb{R}) \) such that \( u = 1/x \) on \((0, \infty)\) and \( u = 0 \) on \((-\infty, 0)\) (Problem 1.4) (9/25/2020)

An example of a distribution \( u \in \mathcal{D}'(\mathbb{R}) \) such that \( u = 1/x \) on \((0, \infty)\) and \( u = 0 \) on \((-\infty, 0)\) is:

\[
\langle u, \phi \rangle = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \frac{\phi(x) - \phi(0)}{x} \, dx.
\]

It obviously has the desired properties and it’s obvious that the limit here exists (since the integrand tends to \( \phi'(0) \) as \( x \to 0^+ \)). To see that it’s a distribution, take any compact \( K \subseteq \mathbb{R} \) and any \( b > 0 \) such that \( K \subseteq [-b, b] \). Then taking any \( \phi \in C^\infty_c(K) \) and using the equation \( \phi(x) = \phi(0) + x \int_{0}^{1} \phi'(tx) \, dt \), we can rewrite the above quantity as:

\[
\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{b} \frac{\phi(x) - \phi(0)}{x} \, dx = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \phi'(tx) \, dx = \int_{0}^{1} \phi'(tx) \, dx,
\]
and so we get the distribution “seminorm estimate” \(|(u, \phi)| \leq b \sup |\phi'|\). It’s interesting to note that by the mathematics in the section discussing the principle value distribution \(x^{-1}\), it’s easy to see that this distribution has order 1. Gunther Uhlmann also observed that if we want to solve the same problem but instead require that \(u = 1/x^n\) on \((0, \infty)\), then we can use:

\[
(u, \phi) = \lim_{\epsilon \to 0^+} \int_\epsilon^{\infty} \phi(x) - \sum_{k=0}^{n-1} (\phi^{(k)}(0)/k!) x^k \frac{\partial^k \phi}{\partial x^k} dx.
\]

The inner sum is of course just the \((n - 1)\)th Taylor polynomial of \(\phi\) based at 0. I wonder what the order of this distribution is?

**An Interesting Example of a Non-Extendible Distribution (Problem 1.5) [2/26/2021].**

Consider the linear form \(u : C_c^\infty(0, \infty) \to \mathbb{C}\) given by

\[
(u, \phi) = \sum_{k=1}^{\infty} \partial^k \phi(1/k).
\]

The claim is that this is a distribution and that it cannot be extended to all of \(\mathbb{R}\) (i.e. there does not exist a \(v \in \mathcal{D}'(\mathbb{R})\) such that the restriction of \(v\) to \((0, \infty)\) is \(u\)). First let’s show that it’s a distribution. Take any compact subset \(K \subseteq (0, \infty)\) of \((0, \infty)\). Since \(K\) is compact we have that the sequence \(1/k\) for \(k \in \mathbb{Z}_+\) escapes \(K\) eventually. More precisely this means that there exists an \(m \in \mathbb{Z}_+\) such that for any integer \(k > m\), \(1/k \notin K\). Thus over \(K\) we have that \(u\) is given by the finite sum

\[
(u, \phi) = \sum_{k=1}^{m} \partial^k \phi(1/k) \quad \forall \phi \in C_c^\infty(K).
\]

Thus \(u\) satisfies the following distribution “semi-norm” estimate over \(K\):

\[
|(u, \phi)| \leq \sum_{k=1}^{m} \sup |\partial^k \phi| \quad \forall \phi \in C_c^\infty(K).
\]

So \(u\) is indeed a distribution. Next let’s prove that \(u\) cannot be extended to all of \(\mathbb{R}\). Let’s prove this by contradiction. Suppose not. Then there exists a distribution \(v \in \mathcal{D}'(\mathbb{R})\) such that \(v|_{(0, \infty)} = u\). Take the compact subset \(K = [-1, 1]\) of \(\mathbb{R}\) and let \(C, N > 0\) be such that

\[
|(v, \phi)| \leq C \sum_{k=0}^{N} \sup |\partial^k \phi| \quad \forall \phi \in C_c^\infty(K).
\]

Let \(\psi \in C_c^\infty(\mathbb{R})\) be a smooth bump function that’s identically one on a neighborhood of zero and whose support is contained in \(K = [-1, 1]\). For any \(m \in \mathbb{Z}_+: m \geq 2\), let \(\phi_m \in C_c^\infty(\mathbb{R})\) be the test function

\[
\phi_m(x) = \psi(2m(m + 1)x) \cdot x^m.
\]
(the condition $m \geq 2$ is for convenience so that as explained below $\text{supp} \phi_m \subseteq K$). Graphically speaking, this test function is equal to $x^m$ in a neighborhood of $1/m$ and whose compact support is contained in an interval that’s situated between $1/(m + 1)$ and $1/(m - 1)$ (not including these two points):

For those interested for the more precise statement, the support of $\phi_m$ is contained in the closed interval centered at $1/m$ with radius half the distance from $1/(m + 1)$ and $1/m$. On one hand we have that (in the last inequality here I bound $x$ by $1/(m - 1)$ since $\phi_m \equiv 0$ past $x > 1/(m - 1)$)

$$|\langle v, \phi_m \rangle| \leq C \sum_{k=0}^{N} \sup|\partial^k \phi_m| \leq C \sum_{k=0}^{N} \sum_{j=0}^{k} \binom{k}{j} \sup|\partial^j [\psi(2m(m + 1)x)]| \cdot \sup|\partial^{k-j}(x^m)|$$

$$\leq C \sum_{k=0}^{N} \sum_{j=0}^{k} \binom{k}{j} \sup|\partial^j \psi| \left(2m(m + 1)^j \cdot \ldots \cdot (m - j + 1) \left(\frac{1}{m - 1}\right)^{m-j}\right).$$

By choosing a big enough constant $D > 0$, we can estimate this bound further by

$$|\langle v, \phi_m \rangle| \leq D \frac{m^N (m + 1)^N m^{N-1}}{(m - 1)^{m-N}}.$$

On the other hand, since $\text{supp} \phi_m \subseteq (0, \infty)$ we have that:

$$\langle v, \phi_m \rangle = \langle u, \phi_m |_{(0, \infty)} \rangle = \partial^m \phi_m (1/m) = m!$$

But we then have a contradiction since the previous inequality implies that $\langle v, \phi_m \rangle \to 0$ as $m \to \infty$ while the above equation implies that $\langle v, \phi_m \rangle \to \infty$ as $m \to \infty$. So indeed no such extension of $u$ can exist.

**Chapter 5**

**Convolution Equations on Forward Cones (Problem 5.5) (1/2/2021)**

*Note:* Here I will write the components of my points/vectors as superscripts. For example, a point $x \in \mathbb{R}^n$ will be explicitly written out as $x = (x^1, \ldots, x^n)$.

Here I discuss the following result which appears as a problem in the book.
Theorem: Let \( n \geq 2 \) be an integer and for any point \( x \in \mathbb{R}^n \) let \( \tilde{x} \) denote the point obtained by projecting it down to its first \( n-1 \) components: \( \tilde{x} = (x^1, \ldots, x^{n-1}) \). In addition, let \( \Gamma \) be the forward cone:

\[
\Gamma = \{ x \in \mathbb{R}^n : x^n \geq c|\tilde{x}| \}
\]

where \( c > 0 \) is some fixed positive constant. Let \( \mathcal{D}'_\Gamma(\mathbb{R}^n) \) denote the set of all distributions over \( \mathbb{R}^n \) whose support is contained in \( \Gamma \):

\[
\mathcal{D}'_\Gamma(\mathbb{R}^n) = \{ u \in \mathcal{D}'(\mathbb{R}^n) : \text{supp } u \subseteq \Gamma \}.
\]

Then the following are true:

a) If \( u_1, \ldots, u_m \in \mathcal{D}'_\Gamma(\mathbb{R}^n) \), then the convolution \( u_1 * \ldots * u_m \) is well defined.

b) If \( u_1, \ldots, u_m \in \mathcal{D}'_\Gamma(\mathbb{R}^n) \) and \( v \in \mathcal{D}'(\mathbb{R}^n) \) is such that \( \text{supp } v \subseteq \{ x^n \geq a \} \) for some fixed \( a \in \mathbb{R} \), then the convolution \( u_1 * \ldots * u_m * v \) is also well defined.

c) Suppose that \( k \in \mathcal{D}'(\mathbb{R}^n) \) is a convolution operator that has a fundamental solution \( E \in \mathcal{D}'_\Gamma(\mathbb{R}^n) \). Then for any \( v \in \mathcal{D}'(\mathbb{R}^n) \) such that \( \text{supp } v \subseteq \{ x^n \geq a \} \), there exists a unique solution \( u \) to the equation \( k * u = v \) such that \( \text{supp } u \subseteq \{ x^n \geq a \} \) as well. Furthermore, for any such solution \( u \), \( \text{supp}(u) \leq \text{supp } E + \text{supp } v \).

Proof: Let’s start with (a). Take any \( u_1, \ldots, u_m \in \mathcal{D}'_\Gamma(\mathbb{R}^n) \). We need to show that the addition function is proper over \( \text{supp } u_1 \times \ldots \times \text{supp } u_m \). We will do this by using the criterion for such properness described in Definition 5.3.1 in the book. Take any \( \delta > 0 \) and suppose that \( x_1, \ldots, x_m \in \mathbb{R}^n \) are such that each \( x_i \in \text{supp } u_i \) and \( |x_1 + \ldots + x_m| \leq \delta \). Since each \( \text{supp } u_i \subseteq \Gamma \), we have that each \( x_i^n \geq 0 \) and so the previous inequality implies that each \( x_i^n \leq \delta \). By definition of \( \Gamma \) we then get that each \( |x_i| \leq \delta/c \) and so each:

\[
|x_i| \leq \sqrt{(\delta/c)^2 + \delta^2} = \delta \sqrt{(1/c)^2 + 1}.
\]

Setting \( \delta' = \delta \sqrt{(1/c)^2 + 1} \) in Definition 5.3.1 then proves the desired properness. This proves (a).

Onwards to (b)! This is proved similarly: take any \( u_1, \ldots, u_m \in \mathcal{D}'_\Gamma(\mathbb{R}^n) \) and any \( v \in \mathcal{D}'(\mathbb{R}^n) \) such that \( \text{supp } v \subseteq \{ x^n \geq a \} \) for some fixed \( a \in \mathbb{R} \). Take any \( \delta > 0 \) and suppose that \( x_i \in \text{supp } u_i \) and \( y \in \text{supp } v \) are such that \( |x_1 + \ldots + x_m + y| \leq \delta \). The last inequality implies that both \( |\tilde{x}_1 + \ldots + \tilde{x}_m + \tilde{y}| \leq \delta \) and \( |x_1^n + \ldots + x_m^n + y^n| \leq \delta \). The latter coupled with the facts that each \( x_i^n \geq 0 \) and \( y^n \geq a \) gives:

\[
|x_i^n| \leq x_1^n + \ldots + x_m^n + a - a \leq x_1^n + \ldots + x_m^n + y^n - a \leq \delta - a,
\]

\[
|y^n| \leq |x_1^n + \ldots + x_m^n + y^n| + |x_1^n + \ldots + x_m^n| \leq \delta + m(\delta - a).
\]

By the definition of \( \Gamma \) we have that each \( |\tilde{x}_i| \leq (\delta - a)/c \). The inequality \( |\tilde{x}_1 + \ldots + \tilde{x}_m + \tilde{y}| \leq \delta \) then gives us that:

\[
|\tilde{y}| \leq |\tilde{x}_1 + \ldots + \tilde{x}_m + \tilde{y}| + |\tilde{x}_1 + \ldots + \tilde{x}_m| \leq \delta + m(\delta - a)/c.
\]
In total we get the results:

\[ |x_i| \leq \sqrt{[(\delta - a)/c]^2 + (\delta - a)^2}, \]
\[ |y| \leq \sqrt{[\delta + m(\delta - a)/c]^2 + [\delta + m(\delta - a)]^2}. \]

Setting \( \delta' > 0 \) in Definition 5.3.1 to be any constant bigger than the two constants on the right-hand sides above then proves (b).

Finally we come to (c). Take any \( v \in \mathcal{D}'(\mathbb{R}^n) \) such that \( \text{supp } v \subseteq \{ x^n \geq a \} \) for some fixed \( a \in \mathbb{R} \). Obviously \( u = E * v \) is a desired solution since by Theorem 5.3.2 (iii) in the book,

\[ \text{supp } u \subseteq \text{supp } E + \text{supp } v \subseteq \{ x^n \geq a \} \]

and

\[ k * u = k * E * v = \delta * v = v \]

(all of these convolutions make sense by part (b)). To prove that this is the unique desired solution, suppose that \( \tilde{u} \) is another such solution. Convolving both sides of the equation \( k * \tilde{u} = v \) by \( E \) from the left gives:

\[ E * k * \tilde{u} = E * v. \]

The left-hand side here is equal to:

\[ E * k * \tilde{u} = k * E * \tilde{u} = \delta * \tilde{u} = \tilde{u}. \]

Plugging this into the previous equation recovers our previous solution: \( \tilde{u} = E * v \). So indeed the desired solution to our equation is unique.

\[ \square \]

Chapter 6


Note: I plan to do much needed maintenance on this entry.

Note: This entry is hard.

Here I give my version of a proof to the Schwartz Kernel Theorem. It’s exactly the same as the proof in this book except that I fill in all of the details. Since this proof involves a lot of steps, I think the best approach for any reader would be to first read Friedlander’s shorter proof in order to get the main idea, and then read this proof if they want to see the details filled in.

Before we prove the theorem, let’s first observe a lemma:

**Lemma:** Suppose that \( f \in C^\infty(\mathbb{R}^n) \) is a smooth function such that it and all of its partials are \( T \)-periodic where \( T > 0 \). Then its Fourier series converges uniformly to \( f \):
\[ f(x) = \sum_{g \in \mathbb{Z}} \hat{f}_g e^{2\pi i (x \cdot g) / T} \]

where each

\[ \hat{f}_g = \frac{1}{T^n} \int_{R_T} f(s) e^{-2\pi i (s \cdot g) / T} ds, \]

where \( R_T \) is any \( T \)-periodic box (for example: \( R_T = [0, T]^n \)). Furthermore, for any \( \alpha \in \mathcal{I}(n) \) (see “notations and conventions”), the \( \partial^\alpha \) partial of the Fourier series’ partial sums converge uniformly to \( \partial^\alpha f \).

**Proof:** I’d say that this is a rather standard fact from the theory of Fourier series. I proved the special case \( T = 1 \) in an earlier theorem in my handwritten notes, and this version is proved exactly the same way.

Now for the Schwartz Kernel Theorem:

**Schwartz Kernel Theorem:** Suppose that \( X \subseteq \mathbb{R}^m \) and \( Y \subseteq \mathbb{R}^n \) are open subsets. A linear map \( \mu : C_c^\infty(Y) \rightarrow \mathcal{D}'(X) \) is sequentially continuous if and only if it’s generated by a Schwartz Kernel \( k \in \mathcal{D}'(X \times Y) \):

\[ \langle \mu \psi, \phi \rangle = \langle k, \phi \otimes \psi \rangle. \]

**Proof:** We already proved in the book that if such a map \( \mu \) is generated by a Schwartz kernel \( k \in \mathcal{D}'(X \times Y) \), then it’s sequentially continuous. So let’s prove the forward implication. The fact that \( k \) is uniquely determined by \( \mu \) is obvious since the above equation determines what \( k \) is equal to on a dense subset of \( C_c^\infty(X \times Y) \) (dense with respect to convergence in \( C_c^\infty(X \times Y) \) of course). So let’s prove the existence of such a \( k \in \mathcal{D}'(X \times Y) \).

Let \( \{K_j\}_{j=1}^\infty \) and \( \{Q_j\}_{j=1}^\infty \) be compact exhaustions of \( X \) and \( Y \) respectively (we will need that fact that each \( K_j \subseteq K_{j+1}^{int} \) and \( Q_j \subseteq Q_{j+1}^{int} \) later). Consider the bilinear form \( B : C_c^\infty(X) \times C_c^\infty(Y) \rightarrow \mathbb{C} \) given by:

\[ B(\phi, \psi) = \langle \mu \psi, \phi \rangle. \]

Now, fix any \( j_0 \in \mathbb{Z}_+ \). Take any \( \psi \in C_c^\infty(Q_{j_0}) \). Since \( \mu \psi \in \mathcal{D}'(X) \), we have that there exist \( C_\psi, M_\psi > 0 \) such that

\[ |B(\phi, \psi)| = |\langle \mu \psi, \phi \rangle| \leq C_\psi \sum_{|\alpha| \leq M_\psi} \sup |\partial^\alpha \phi| \quad \forall \phi \in C_c^\infty(K_{j_0}). \]
Now take any \( \phi \in C_c^\infty(K_{j_0}) \). The map \( \psi \mapsto \mu \psi \) being sequentially continuous implies that the map \( \psi \mapsto \langle \mu \psi, \phi \rangle \) is also sequentially continuous. Thus the latter map is a distribution in \( D'(Y) \) and so there exist \( C_\phi, N_\phi > 0 \) such that:

\[
|B(\phi, \psi)| = |\langle \mu \psi, \phi \rangle| \leq C_\phi \sum_{|\beta| \leq N_\phi} \sup |\partial^\beta \psi| \quad \forall \psi \in C_c^\infty(Q_{j_0}).
\]

These two inequalities show that the restriction \( B_{j_0} : C_c^\infty(K_{j_0}) \times C_c^\infty(Q_{j_0}) \to \mathbb{C} \) of \( B \) is a separately continuous bilinear form over the Fréchet spaces \( C_c^\infty(K_{j_0}) \) and \( C_c^\infty(Q_{j_0}) \). Now, noting that Fréchet spaces are also Banach spaces we have by a quick corollary of the uniform boundedness principle that \( B_{j_0} \) is continuous with respect to the product topology. I prove this corollary in my electronic diary about Folland’s “Real Analysis” book. In the that same electronic diary, I also prove an equivalent condition for a bilinear map between Fréchet spaces being continuous which in our case gives that there exist \( C_0, N_0 > 0 \) such that:

\[
|B(\phi, \psi)| = |B_{j_0}(\phi, \psi)| \leq C_0 \left( \sum_{|\alpha| \leq N_0} \sup |\partial^\alpha \phi| \right) \left( \sum_{|\beta| \leq N_0} \sup |\partial^\beta \psi| \right)
\]

\[
\forall \phi \in C_c^\infty(K_{j_0}) \quad \text{and} \quad \forall \psi \in C_c^\infty(Q_{j_0}).
\]

Ok, with this in hand we are now ready to begin constructing the \( k \in D'(X \times Y) \) that we want. We will do this by defining continuous linear forms \( k_j : C_c^\infty(K_j \times Q_j) \to \mathbb{C} \) for \( j \in \mathbb{Z}_+ \) and then use their values to define \( k \). Fix any \( j_0 \in \mathbb{Z}_+ \). Let \( \rho \in C_c^\infty(K_{j_0+1}) \) and \( \sigma \in C_c^\infty(Q_{j_0+1}) \) be such that \( \rho \equiv 1 \) and \( \sigma \equiv 1 \) on neighborhoods of \( K_{j_0} \) and \( Q_{j_0} \) respectively. Let \( b > 0 \) be such that \( K_{j_0+1} \times Q_{j_0+1} \subseteq [-b, b]^{m+n} \). Now, take any \( \chi \in C_c^\infty(K_{j_0} \times Q_{j_0}) \). Define the value of \( k_{j_0} \) at \( \chi \) to be:

\[
\langle k_{j_0}, \chi \rangle = \sum_{(g,h) \in \mathbb{Z}^{m+n}} B(\hat{\chi}(g,h)\rho(x)e^{2\pi i(x\cdot g)/(2b)}, \sigma(y)e^{2\pi i(y\cdot h)/(2b)})
\]

where \( \hat{\chi}(g,h) \) are the Fourier coefficients of \( \chi \):

\[
\hat{\chi}(g,h) = \frac{1}{(2b)^{m+n}} \int_{[-b,b]^{m+n}} \chi(x,y)e^{-2\pi i(x\cdot g + y\cdot h)/(2b)} \, dx \, dy.
\]

Technically I should be writing \( \chi_{\mathbb{R}^m \times \mathbb{R}^n} \) (see notations and conventions) rather than \( \chi \) inside the above integral since \( \chi \) is not necessarily defined over \( [-b, b] \). Ok, in order for the previous expression to make sense let’s prove that the sum on the right-hand side converges uniformly. Before we do that though, I’d like to point out that once we prove that the above sum makes sense and is finite, then it will be clear that \( k_{j_0} \) is a linear form since the equation for the Fourier coefficient is linear in \( \chi \) and \( B \) is linear in its first argument.
Ok, let’s prove that that sum is absolutely convergent. First let’s create a bound to estimate how large the above Fourier coefficients of \( \chi \) can get. Fix any index \((g, h) \in \mathbb{Z}^m \times \mathbb{Z}^n\). First a piece of notation: for an any multi-index \( \alpha \in \mathcal{J}(r) \), let \( \mathcal{H}(\alpha) \) be the multi-index whose \( s^\text{th} \) component is equal to 1 if \( \alpha_s > 0 \) and is equal to 0 if \( \alpha_s = 0 \). Let \( N = N_{j_0 + 1} \). Notice that since

\[
supp \chi \subseteq K_{j_0} \times Q_{j_0} \subseteq K_{j_0 + 1} \times Q_{j_0 + 1} \subseteq [-b, b]^{m+n},
\]

we have that \( \chi \) and all of its partials are zero on the boundary of the box \([-b, b]^{m+n}\). So for any \((g, h) \in \mathbb{Z}^m \times \mathbb{Z}^n\) we have by many integrations by parts that

\[
\hat{\chi}(g,h) = \frac{1}{(2b)^{m+n}} \frac{1}{-2\pi i} \prod_{r=1}^m g_r^{N+2} \prod_{s=1}^n h_s^{N+2} \int_{[-b, b]^{m+n}} \partial^{(N+2)\mathcal{H}(g,h)} [\chi(x,y)] e^{-2\pi i (x \cdot g + y \cdot h)/(2b)} dxdy.
\]

Let \( E_1 > 0 \) be a constant such that:

\[
\left| \frac{1}{(2b)^{m+n}} \frac{2b}{-2\pi i} \right|^{(N+2)\mathcal{H}(g,h)} \leq E_1.
\]

Furthermore, notice that we can remove the unpleasant \( g_r \neq 0 \) and \( h_s \neq 0 \) indexing rules above by writing the estimate (here I use the fact that \( 1/r \leq 2/(1 + r) \) if \( r > 0 \) is an integer):

\[
\left| \prod_{r=1}^m g_r^{N+2} \prod_{s=1}^n h_s^{N+2} \right| \leq \prod_{r=1}^m (1 + |g_r|^{N+2}) \prod_{s=1}^n (1 + |h_s|^{N+2})
\]

Setting \( E_2 = 2^{m+n} E_1 \), we get the following estimate on our Fourier coefficient:

\[
|\hat{\chi}(g,h)| \leq E_2 \prod_{r=1}^m (1 + |g_r|^{N+2}) \prod_{s=1}^n (1 + |h_s|^{N+2}) \text{Vol}([-b, b]^{m+n}) \sup |\partial^{(N+2)\mathcal{H}(g,h)} \chi|
\]

Setting \( E_3 = E_2 \text{Vol}([-b, b]^{m+n}) \), notice that we can furthermore estimate our Fourier coefficient as:

\[
|\hat{\chi}(g,h)| \leq E_3 \left( \sum_{|\gamma| \leq N+2} \sup |\partial^\gamma \chi| \right) \prod_{r=1}^m (1 + |g_r|^{N+2}) \prod_{s=1}^n (1 + |h_s|^{N+2})
\]

which is written in a bit more familiar terms. Great! We’re going to use this estimate below.

Meanwhile, going back to our definition of \( (k_{j_0}, \chi) \), notice that we can bound each summand term on the right by:

\[
|B(\hat{\chi}(g,h)\rho(x)e^{2\pi i (x \cdot g)/(2b)}, \sigma(y)e^{2\pi i (y \cdot h)/(2b)})|
\]
\[ \leq C_0 \left( \sum_{|\alpha| \leq N} \sup |\partial^{\alpha} [\hat{\chi}(g,h)\rho(x)e^{2\pi i(x-g)/(2b)}]| \right) \left( \sum_{|\beta| \leq N} \sup |\partial^{\beta} [\sigma(y)e^{2\pi i(y-h)/(2b)}]| \right) \]

Now apply the product rule on each \( \partial^{\alpha} \) and \( \partial^{\beta} \) partials (recall that by convention \( 0^0 = 1 \) when raising something to a multi-index):

\[
|B(\hat{\chi}(g,h)\rho(x)e^{2\pi i(x-g)/(2b)}, \sigma(y)e^{2\pi i(y-h)/(2b)})| \\
\leq C_0 |\hat{\chi}(g,h)| \left( \sum_{|\alpha| \leq N} \sum_{\eta \leq \alpha} \sup \left| \frac{\alpha!}{\eta!(\alpha-\eta)!} \left( \frac{2\pi i}{2b} \right)^{|\eta|} g^\eta e^{2\pi i(x-g)/(2b)} \partial^{\alpha-\eta} \rho \right| \right) \\
\cdot \left( \sum_{|\beta| \leq N} \sum_{v \leq \beta} \sup \left| \frac{\beta!}{v!(\beta-v)!} \left( \frac{2\pi i}{2b} \right)^{|v|} h^v e^{2\pi i(y-h)/(2b)} \partial^{\beta-v} \rho \right| \right) \\
\]

If we distribute above sum, we merely get a big linear combination of terms of the form \( g^\eta h^v \). So for some collection of coefficients \( A_{(\eta,v)} \),

\[
|B(\hat{\chi}(g,h)\rho(x)e^{2\pi i(x-g)/(2b)}, \sigma(y)e^{2\pi i(y-h)/(2b)})| \leq C_0 |\hat{\chi}(g,h)| \sum_{|\eta| \leq N} A_{(\eta,v)} g^\eta h^v. \\
\]

Now, if we plug in the above estimate for \( |\hat{\chi}(g,h)| \) into this inequality we get that for some constant \( E_4 > 0 \),

\[
|B(\hat{\chi}(g,h)\rho(x)e^{2\pi i(x-g)/(2b)}, \sigma(y)e^{2\pi i(y-h)/(2b)})| \\
\leq E_4 \sum_{|\eta| \leq N} \prod_{r=1}^m (1 + |g_r|^N)^2 \prod_{s=1}^n (1 + |h_s|^N)^2 \sum_{|\gamma| \leq N+2} \sup |\partial^{\gamma} \chi| \\
\]

Ok, since each \( |\eta| \leq N \) in the first sum, we have that

\[
\frac{g^\eta}{\prod_{r=1}^m (1 + |g_r|^N)^2} \leq \frac{g^\eta}{\prod_{r=1, g_r \neq 0}^m g_r^{N+2}} \leq \frac{1}{\prod_{r=1, g_r \neq 0}^m g_r^2} \leq \frac{2^m}{\prod_{r=1}^m (1 + |g_r|^2)}. \\
\]

By a similar calculation we have that

\[
\frac{h^\nu}{\prod_{s=1}^n (1 + |h_s|^N)^2} \leq \frac{2^n}{\prod_{s=1}^n (1 + |h_s|^2)}. \\
\]

So we get that for some constant \( E_5 > 0 \),

\[
|B(\hat{\chi}(g,h)\rho(x)e^{2\pi i(x-g)/(2b)}, \sigma(y)e^{2\pi i(y-h)/(2b)})| \\
\]
\[
\leq E_5 \prod_{r=1}^{m} \frac{1}{(1 + |g_r|^2)} \prod_{s=1}^{n} (1 + |h_s|^2) \sum_{|y| \leq N+2} \sup |\partial^y \chi|.
\]

Tracing the above calculation again, notice that the value of the constant \(E_5\) is independent of \((g, h)\) (and even \(\chi\)). So the above estimate holds for all \((g, h) \in \mathbb{Z}^m \times \mathbb{Z}^n\) (and all \(\chi\)). Thus the above inequality shows that that the sum in the definition of \(\langle k_{j_0}, \chi \rangle\) decays like \(g_2^2\) and \(h_2^2\) in all direction. Hence that sum is indeed absolutely convergent and thus makes sense. The above inequality shows more: it gives us the estimate:

\[
|\langle k_{j_0}, \chi \rangle| \leq E_5 \left( \sum_{(g, h) \in \mathbb{Z}^m \times \mathbb{Z}^n} \prod_{r=1}^{m} \frac{1}{(1 + |g_r|^2)} \prod_{s=1}^{n} (1 + |h_s|^2) \right) \sum_{|y| \leq N+2} \sup |\partial^y \chi|.
\]

Again, since the \(\Sigma_{(g, h) \in \mathbb{Z}^m \times \mathbb{Z}^n}\) sum here is finite and the value of the constant \(E_5\) is independent of \(\chi\), this shows that \(k_{j_0}\) is in fact a continuous linear form over \(C^\infty_c(K_{j_0} \times Q_{j_0})\).

Great, let’s show one last property of \(k_{j_0}\). Let’s show that for any \(\phi \in C^\infty_c(K_{j_0})\) and any \(\psi \in C^\infty_c(Q_{j_0})\),

\[
\langle k_{j_0}, \phi \otimes \psi \rangle = B(\phi, \psi) = \langle \mu \psi, \phi \rangle.
\]

The Fourier coefficients of \((\phi \otimes \psi)^{\mathbb{R}^m \times \mathbb{R}^n}\) are given by:

\[
(\phi \otimes \psi)_{(g, h)} = \frac{1}{(2b)^{m+n}} \int_{[-b, b]^m n} \phi(x) \psi(y) e^{-2\pi i (x \cdot g + y \cdot h)/(2b)} dx dy
\]

\[
= \frac{1}{(2b)^m} \int_{[-b, b]^m} \phi(x) e^{-2\pi i (x \cdot g)/(2b)} dx \cdot \frac{1}{(2b)^n} \int_{[-b, b]^n} \psi(y) e^{-2\pi i (y \cdot h)/(2b)} dy.
\]

Setting \(\hat{\phi}_g\) and \(\hat{\psi}_h\) to be the Fourier coefficients of \(\phi^{\mathbb{R}^m}\) and \(\psi^{\mathbb{R}^n}\) respectively:

\[
\hat{\phi}_g = \frac{1}{(2b)^m} \int_{[-b, b]^m} \phi(x) e^{-2\pi i (x \cdot g)/(2b)} dx,
\]

\[
\hat{\psi}_h = \frac{1}{(2b)^n} \int_{[-b, b]^n} \psi(x) e^{-2\pi i (y \cdot h)/(2b)} dy,
\]

we then get the trivial tensor Fourier coefficient relation:

\[
(\phi \otimes \psi)_{(g, h)} = \hat{\phi}_g \cdot \hat{\psi}_h.
\]

Using this in the definition of \(\langle k_{j_0}, \phi \otimes \psi \rangle\) then gives us that

\[
\langle k_{j_0}, \phi \otimes \psi \rangle = \sum_{(g, h) \in \mathbb{Z}^m \times \mathbb{Z}^n} B(\hat{\phi}_g \rho(x) e^{2\pi i (x \cdot g)/(2b)}, \hat{\psi}_h \sigma(y) e^{2\pi i (y \cdot h)/(2b)}).
\]
Now, it’s easy to see by the lemma stated before this theorem that the series
\[
\sum_{g \in \mathbb{Z}^m} \hat{\phi}_g \rho(x)e^{2\pi i(x \cdot g)/(2b)} = \rho(x) \sum_{g \in \mathbb{Z}^m} \hat{\phi}_g e^{2\pi i(x \cdot g)/(2b)}.
\]
converges to \( \rho \cdot \phi \) in \( C^\infty_c(\mathbb{R}^m) \). So this series converges to \( \phi \) in \( C^\infty_c(K_{j_0+1}) \) since \( \rho \cdot \phi = \phi \) on \( K_{j_0+1} \) (recall that \( \text{supp } \phi \subseteq K_{j_0} \) and \( \rho \equiv 1 \) on \( K_{j_0} \)). For the same reason, we have that the series
\[
\sum_{h \in \mathbb{Z}^n} \hat{\psi}_h \sigma(y)e^{2\pi i(y \cdot h)/(2b)}
\]
converges to \( \psi \) in \( C^\infty_c(Q_{j_0+1}) \) as well. Using the fact that \( B_{j_0+1} : C^\infty_c(K_{j_0+1}) \times C^\infty_c(Q_{j_0+1}) \to \mathbb{C} \) is separately continuous we then have that
\[
\langle k_{j_0}, \phi \otimes \psi \rangle = \sum_{g \in \mathbb{Z}^m} \sum_{h \in \mathbb{Z}^n} B_{j_0+1}(\hat{\phi}_g \rho(x)e^{2\pi i(x \cdot g)/(2b)}, \hat{\psi}_h \sigma(y)e^{2\pi i(y \cdot h)/(2b)})
\]
\[
= \sum_{g \in \mathbb{Z}^m} B_{j_0+1}(\hat{\phi}_g \rho(x)e^{2\pi i(x \cdot g)/(2b)}, \psi) = B_{j_0}(\phi, \psi) = B(\phi, \psi) = \langle \mu \psi, \phi \rangle.
\]
We are now finally ready for the last step in the proof of this theorem. Define the function \( k : C^\infty_c(X \times Y) \to \mathbb{C} \) as follows. Take any test function \( \chi \in C^\infty_c(X \times Y) \) and let \( j_0 \in \mathbb{Z}_+ \) be such that \( \text{supp } \chi \subseteq K_{j_0} \times Q_{j_0} \). Then set:
\[
\langle k, \chi \rangle = \langle k_{j_0}, \chi \rangle.
\]
Let’s prove that this is well defined by showing that this is independent of the \( j_0 \) that we chose that satisfies the above property. Let \( r_0 \in \mathbb{Z}_+ \) be any other such index and let’s suppose without loss of generality that \( j_0 \leq r_0 \). As before, let \( \rho \in C^\infty_c(K_{j_0+1}) \) and \( \sigma \in C^\infty_c(Q_{j_0+1}) \) be such that \( \rho \equiv 1 \) and \( \sigma \equiv 1 \) on neighborhoods of \( K_{j_0} \) and \( Q_{j_0} \) respectively. Letting \( b > 0 \) be such that \( K_{j_0+1} \times Q_{j_0+1} \subseteq [-b, b]^{m+n} \) we have by a similar discussion as above that the series
\[
\sum_{(g,h) \in \mathbb{Z}^m \times \mathbb{Z}^n} \hat{\chi}_{(g,h)} \rho(x)e^{2\pi i(x \cdot g)/(2b)} \sigma(y)e^{2\pi i(y \cdot h)/(2b)}
\]
converges to \( \chi \) in \( C^\infty_c(K_{j_0+1} \times Q_{j_0+1}) \) and thus in \( C^\infty_c(K_{r_0+1} \times Q_{r_0+1}) \) (since \( K_{j_0+1} \times Q_{j_0+1} \subseteq K_{r_0+1} \times Q_{r_0+1} \)). The continuity and linearity of both \( k_{j_0} \) and \( k_{r_0} \) implies that their values at \( \chi \) are given by:
\[
\langle k_{j_0}, \chi \rangle = \sum_{(g,h) \in \mathbb{Z}^m \times \mathbb{Z}^n} k_{j_0}(\hat{\chi}_{(g,h)} \rho(x)e^{2\pi i(x \cdot g)/(2b)} \otimes \sigma(y)e^{2\pi i(y \cdot h)/(2b)}),
\]
\[
\langle k_{r_0}, \chi \rangle = \sum_{(g,h) \in \mathbb{Z}^m \times \mathbb{Z}^n} k_{r_0}(\hat{\chi}_{(g,h)} \rho(x)e^{2\pi i(x \cdot g)/(2b)} \otimes \sigma(y)e^{2\pi i(y \cdot h)/(2b)}),
\]
both of which are equal to
\[
\sum_{(g,h)\in \mathbb{Z}^m \times \mathbb{Z}^n} B\left(\hat{\chi}(g,h)\rho(x)e^{2\pi i (x \cdot g)/(2b)}, \sigma(y)e^{2\pi i (y \cdot h)/(2b)}\right).
\]

So \(\langle k_{j_0}, \chi \rangle = \langle k_{j_0'}, \chi \rangle\), and thus \(k\) is indeed well defined. It’s easy to see from the definition of \(k\) that it’s linear. To see that it’s a distribution, we also have to show that it satisfies the distribution “seminorm” like inequalities. Take any compact set \(R \subseteq X \times Y\). Let \(j_0 \in \mathbb{Z}_+\) be such that \(R \subseteq K_{j_0} \times Q_{j_0}\). Then since \(k_{j_0}\) is continuous we have that there exist \(C, N > 0\) such that for any \(\chi \in C_c^\infty(R) \subseteq C_c^\infty(K_{j_0} \times Q_{j_0})\),

\[
|\langle k, \chi \rangle| = |\langle k_{j_0}, \chi \rangle| \leq C \sum_{|y| \leq N} \sup |\partial^y \chi|.
\]

So \(k\) is indeed a distribution over \(X \times Y\) (i.e. \(k \in \mathcal{D}'(X \times Y)\)). The last thing to show is that it satisfies the property that we desire. Take any \(\phi \in C_c^\infty(X)\) and any \(\psi \in C_c^\infty(Y)\). Let \(j_0 \in \mathbb{Z}_+\) be such that \(\text{supp } \phi \times \text{supp } \psi \subseteq K_{j_0} \times Q_{j_0}\). Then we have that:

\[
\langle k, \phi \otimes \psi \rangle = \langle k_{j_0}, \phi \otimes \psi \rangle = \langle \mu \psi, \phi \rangle.
\]

So \(k\) is the Schwartz kernel of our map \(\mu\). With this we’ve proven the theorem.

\[\blacksquare\]

**Chapter 8**

**Structure Theorem for Tempered Distributions [Theorem 8.3.1] (5/11/2021)**

In this entry I work through the proof of the structure theorem for tempered distributions (Theorem 8.3.1 in the book) by putting it into my own words and filling in the details.

**Theorem:** A distribution is a tempered distribution if and only if it is the derivative of a continuous function of polynomial growth.

**Proof:** First suppose that a distribution \(u \in \mathcal{D}'(\mathbb{R}^n)\) is the derivative of a continuous function \(f \in C^0(\mathbb{R}^n)\) of polynomial growth:

\[u = \partial^\alpha f,\]

where \(\alpha \in \mathcal{I}(n)\) of course. We want to show that \(u\) is a tempered distribution. Since \(f\) is of polynomial growth, there exist \(C > 0\) and \(M \in \mathbb{Z}_+\) such that \(|f(x)| \leq C(1 + |x|)^M\). Now, I claim that the linear functional \(S(\mathbb{R}^n) \to \mathbb{C}\) given by

(Eq 1) \[\phi \mapsto (-1)^{|\alpha|} \int f \partial^\alpha \phi\]

is a well-defined continuous extension of the distribution \(\partial^\alpha f : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}\) to \(S(\mathbb{R}^n)\). If we prove this, then this will show that \(u\) is indeed a tempered distribution. Take any \(\phi \in S(\mathbb{R}^n)\).
First let’s show that the above integral even exists. We estimate (here $m$ is the Lebesgue measure)

$$\int |f \partial^a \phi| \leq C \int (1 + |x|)^M |\partial^a \phi| = C \int_{B_1(0)} (1 + |x|)^M |\partial^a \phi| + \int_{B_1(0)^c} \frac{(1 + |x|)^M |x|^{2n} |\partial^a \phi|}{|x|^{2n}}$$

$$\leq C \left[ m(B_1(0)) \sup\{(1 + |x|)^M |\partial^a \phi|\} + \int_{B_1(0)^c} \frac{1}{|x|^{2n}} dx \cdot \sup\{(1 + |x|)^M |x|^{2n} |\partial^a \phi|\} \right].$$

The last quantity is finite since $\phi \in S(\mathbb{R}^n)$ and thus the integral in (Eq 1) is indeed well defined. It’s clear that (Eq 1) extends $\partial^a f : C^\infty_c(\mathbb{R}^n) \to \mathbb{C}$ to $S(\mathbb{R}^n)$. So all that’s left to prove is that the linear functional defined by (Eq 1) is also continuous. By exactly the same computation as above, for some constant $C_2 > 0$ we obtain the estimate:

$$\left|(-1)^{|a|} \int f \partial^a \phi\right| \leq C_2 [\sup\{(1 + |x|)^M |\partial^a \phi|\} + \sup\{(1 + |x|)^M |x|^{2n} |\partial^a \phi|\}].$$

By expanding the $(1 + |x|)^M$ via the binomial theorem and then doing some further estimates, we see that the above inequality implies that there exist $C_3 > 0$ and $N \in \mathbb{Z}_+$ such that

$$\left|(-1)^{|a|} \int f \partial^a \phi\right| \leq C_3 \sum_{|\beta| \leq N} \sup |x^\beta \partial^a \phi|.$$ 

Thus the linear functional defined by (Eq 1) is indeed continuous. As discussed above, this proves that $u$ is a tempered distribution.

Now let’s prove the other direction. Suppose that $u \in \mathcal{D}'(\mathbb{R}^n)$ is a tempered distribution. We will assume that the support of $u$ is contained in $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : \text{each } x_k > 0 \}$. The general case will then follow from the easy-to-see fact that any tempered distribution can be represented as a finite sum of translations and reflections of such tempered distributions and that reflections and translations of functions of polynomial growth are still of polynomial growth.

Alright, since $u$ is a tempered distribution we have that there exist $C > 0$ and $N \in \mathbb{Z}_+$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|a|, |\beta| \leq N} \sup |x^a \partial^\beta \phi| \quad \forall \phi \in S(\mathbb{R}^n).$$

Let $E_{N+2} : \mathbb{R}^n \to \mathbb{C}$ denote the function

$$E_{N+2}(x) = \begin{cases} \frac{(x_1)^{N+1} \cdots (x_n)^{N+1}}{[(N + 1)!]^n} & \text{if each } x_k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Observe that $E_{N+2}$ is in $C^N(\mathbb{R}^n)$ and that

$$\partial^{(N+2)} E_{N+2} = \delta.$$
in the sense of distributions (here $\bar{1} \in J(n)$ denotes the multi-index with all ones). Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a smooth function such that

1.) $\rho \geq 0$
2.) $\int \rho = 1$
3.) $\text{supp} \rho \subseteq B_1(0)$.

For each $j \in \mathbb{Z}_+$, let $\rho_j \in C_c^\infty(\mathbb{R}^n)$ denote the functions $\rho_j = j^n \rho(jx)$. Observe that $\rho_j \to \delta$ in $D'(\mathbb{R}^n)$. Now, we will prove that $E_{N+2} \ast u$ is a well-defined continuous function of polynomial growth such that

$$\partial^{(N+2)\bar{1}}(E_{N+2} \ast u) = u.$$ 

The fact that $E_{N+2} \ast u$ is well defined comes from the fact that the addition function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(x, y) \mapsto x + y$ is clearly proper on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ and hence proper on $\text{supp} \ E_{N+2} \times \text{supp} \ u$ (recall that $\text{supp} \ E_{N+2}$ and $\text{supp} \ u$ are both contained in $\mathbb{R}^n_+$). The fact that the above equation holds follows immediately from

$$\partial^{(N+2)\bar{1}}(E_{N+2} \ast u) = \partial^{(N+2)\bar{1}}(E_{N+2}) \ast u = \delta \ast u = u.$$ 

Next let’s show that $E_{N+2} \ast u$ is a continuous function. We will do this by showing that it’s the uniform limit over compact sets of the continuous functions described in the following claim.

**Claim:** For any $j \in \mathbb{Z}_+$, the following is a well-defined smooth function over $\mathbb{R}^n$:

$$E_{N+2} \ast \rho_j \ast u.$$ 

Furthermore, if $\sigma \in C^\infty(\mathbb{R}^n)$ is such that $\sigma \equiv 1$ on a neighborhood of $\text{supp} \ u$ and $\text{supp} \ \sigma \subseteq \mathbb{R}^n_+$, then this function is explicitly given by

(Eq 2) $$E_{N+2} \ast \rho_j \ast u(x) = (u(y), \sigma(y)(E_{N+2} \ast \rho_j)(x - y)).$$

**Proof:** Fix any $j \in \mathbb{Z}_+$. The fact that the convolution $E_{N+2} \ast \rho_j \ast u$ is well defined follows from the not-hard-to-prove fact that the restriction of the addition function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(x, y, z) \mapsto x + y + z$ to $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$ is proper (recall that $\text{supp} \ \rho_j \subseteq B_{1/j}(0)$). Now, let’s see why the right-hand side of the above equation even makes sense. Specifically we have to show that argument of $u$ on the right-hand side (i.e. $\sigma(y)(E_{N+2} \ast \rho_j)(x - y)$) is smooth and of compact support. The compactness of its support follows from:

$$\sigma(y)(E_{N+2} \ast \rho_j)(x - y) \neq 0 \quad \Rightarrow \quad y \in \text{supp} \ \sigma \quad \text{and} \quad (x - y) \in \text{supp} \ E_{n+2} + \text{supp} \rho_j$$

$$\Rightarrow \quad y \in \mathbb{R}^n_+ \quad \text{and} \quad (x - y) \in \mathbb{R}^n_+ + B_{1/j}(0) \quad \Rightarrow \quad y \in \mathbb{R}^n_+ \quad \text{and} \quad y \in x - \mathbb{R}^n_+ + B_{1/j}(0)$$

$$\Rightarrow \quad |y| \leq |x| + 1/j.$$ 

The smoothness of the right-hand side of (Eq 2) also follows from this and Theorem 4.1.1 in the book.
To finish proving the claim, all we have to do now is show that equality holds in (Eq 2). To start, let \( f = E_{N+2} \ast \rho_j \). Let \( \sigma_f, \sigma_u \in C^\infty(\mathbb{R}^n) \) be such that

1. \( \sigma_f \equiv 1 \) on a neighborhood of \( \text{supp}(f) \) and \( \text{supp} \sigma_f \subseteq \text{supp}(f) + B_1(0) \),
2. \( \sigma_u \equiv 1 \) on a neighborhood of \( \text{supp} u \) and \( \text{supp} \sigma_u \subseteq \text{supp} u + B_1(0) \).

Take any test function \( \phi \in C_c^\infty(\mathbb{R}^n) \). Analogously, let \( \sigma_\phi \in C_c^\infty(\mathbb{R}^n) \) be such that \( \sigma_\phi \equiv 1 \) on a neighborhood of \( \text{supp} \phi \) (note the requirement for \( \text{supp} \sigma_\phi \) to be compact). By definition, we then have that

\[
\langle f \ast u(z), \phi(z) \rangle = \langle f(x) \otimes u(y), \sigma_f(x)\sigma_u(y)\phi(x + y) \rangle
\]

\[
= \langle u(y), \int f(x)\sigma_f(x)\sigma_u(y)\phi(x + y)dx \rangle = \langle u(y), \int f(z - y)\sigma_u(y)\sigma_\phi(z)\phi(z)dx \rangle
\]

\[
= \langle u(y), \langle \phi(z), \sigma_u(y)\sigma_\phi(z)f(z - y) \rangle \rangle = \langle u(y) \otimes \phi(z), \sigma_u(y)\sigma_\phi(z)f(z - y) \rangle
\]

\[
= \int \phi(z)(u(y), \sigma_u(y)\sigma_\phi(z)f(z - y))dz = \langle u(y), \sigma_u(y)f(z - y) \rangle, \phi(z) \rangle.
\]

Since \( \phi \in C_c^\infty(\mathbb{R}^n) \) was chosen arbitrarily, this shows that

\[
f \ast u(z) = \langle u(y), \sigma_u(y)f(z - y) \rangle.
\]

From here (Eq 2) follows immediately.

End of Proof of Claim.

Let \( \sigma \in C^\infty(\mathbb{R}^n) \) be as in the above claim. First let’s show that the sequence of functions \( E_{N+2} \ast \rho_j \ast u \) is uniformly Cauchy over compact sets and thus converges to some continuous function. Fix any compact subset \( K \subseteq \mathbb{R}^n \) and let \( R > 0 \) be such that \( K \subseteq B_R(0) \). Then, by the above claim we have that

(Eq 3)

\[
\sup_{x \in K} \left| E_{N+2} \ast \rho_k \ast u(x) - E_{N+2} \ast \rho_j \ast u(x) \right|
\]

\[
= \sup_{x \in K} \left| \langle u(y), \sigma(y) \rangle \left( E_{N+2} \ast \rho_k(x - y) - E_{N+2} \ast \rho_j(x - y) \right) \right|
\]

\[
\leq c \sum_{|a|,|b| \leq N} \sup_{x \in K} \sup_{y \in \mathbb{R}^n} \left| y^a \partial^b \sigma(y) \left( E_{N+2} \ast \rho_k(x - y) - E_{N+2} \ast \rho_j(x - y) \right) \right|.
\]

It’s not hard to see that there exists a compact subset \( Q \subseteq \mathbb{R}^n \) such that for any fixed \( x \in K \), the support of the function

\[
\sigma(y) \left( E_{N+2} \ast \rho_k(x - y) - E_{N+2} \ast \rho_j(x - y) \right)
\]

is contained in \( Q \) (for instance, \( Q = B_{R+1}(0) \) will work). Because of this, we can change the “\( y \in \mathbb{R}^n \)” to “\( y \in Q \)” in the last supremum above without changing the value of the supremum.
It’s not hard to see then by the product rule that for some \( \tilde{C} > 0 \) the last quantity in (Eq 3) is further bounded by

\[
\tilde{C} \sum_{|y| \leq N} \sup_{x \in K} |\partial_y^1 [E_{N+2} * \rho_k(x - y) - E_{N+2} * \rho_j(x - y)]| \leq \tilde{C} \sum_{|y| \leq N} \sup_{x \in K-Q} |\partial_y^1 [E_{N+2} * \rho_k(z) - E_{N+2} * \rho_j(z)]|.
\]

Since \( E_{N+2} \in C^N(\mathbb{R}^n) \), it’s well known that for \( \gamma \in \mathcal{J}(n) \) such that \( |\gamma| \leq N \), \( \partial^\gamma (E_{N+2} * \rho_k) \) converges uniformly to \( \partial^\gamma E_{N+2} \) over compact sets (this is a slight variant of Theorem 1.2.1 in the book). Thus we see that the above quantity goes to zero as \( k, j \to \infty \). So the first quantity in (Eq 3) also goes to zero as \( k, j \to \infty \):

\[
\sup_{x \in K} |E_{N+2} * \rho_k * u(x) - E_{N+2} * \rho_j * u(x)| \to 0 \quad \text{as} \quad k, j \to \infty.
\]

So indeed \( E_{N+2} * \rho_j * u \) are uniformly Cauchy over compact sets and thus converge pointwise to some continuous function. I claim that that continuous function that they converge to is \( E_{N+2} * u \). To see this, take any test function \( \phi \in C_c^n(\mathbb{R}^n) \) and do (here I interchange “lim” and “∫” which I can do because of uniform convergence over compact sets)

\[
\langle \lim_{j \to \infty} (E_{N+2} * \rho_j * u), \phi \rangle = \int \lim_{j \to \infty} (E_{N+2} * \rho_j * u(x)) \phi(x)dx = \lim_{j \to \infty} \int E_{N+2} * \rho_j * u(x) \phi(x)dx = \lim_{j \to \infty} \langle E_{N+2} * \rho_j * u, \phi \rangle = \langle E_{N+2} * \delta * u, \phi \rangle = \langle E_{N+2} * u, \phi \rangle.
\]

Hence \( E_{N+2} * u \) is indeed equal to the function that is the pointwise limit of \( E_{N+2} * \rho_j * u \) and thus is a continuous function.

The last thing left to do is to show that \( E_{N+2} * u \) is of polynomial growth. This is again just a game of bounding things. For any \( x \in \mathbb{R}^n \) we have that

\[
|E_{N+2} * u(x)| = \lim_{j \to \infty} |E_{N+2} * \rho_j * u(x)| = \lim_{j \to \infty} |\langle u(y), \sigma(y) \cdot E_{N+2} * \rho_j(x - y) \rangle| \leq C \sum_{|\alpha| + |\beta| \leq N} \lim_{j \to \infty} \sup_{y \in \mathbb{R}^n} |\gamma^\alpha \beta_y^\beta (\sigma(y) \cdot E_{N+2} * \rho_j(x - y))|
\]

As before, it’s not hard to see that the support of the function \( \sigma(y) \cdot E_{N+2} * \rho_j(x - y) \) as a function of \( y \) is contained in \( \mathbb{R}^n_+ \cap B_{|x|+1}(0) \). Thus we can change the “\( y \in \mathbb{R}^n \)” to “\( y \in \mathbb{R}^n_+ \cap B_{|x|+1}(0) \)” in the above supremum without changing its value. By the product rule, it’s then not hard to see that for some \( C_2 > 0 \) the above quantity can further be bounded by
Now, we have that $|y^{\alpha}| \leq (|x| + 1)^{|\alpha|}$ for any $y$ in the domain of the above supremum. In addition, as before, for any $\beta \in I(n)$ such that $|\beta| \leq N$ we have that $\partial^\beta(E_{N+2} \ast \rho_j)$ converges to $\partial^\beta E_{N+2}$ uniformly over compact sets and so we can interchange the “limit” and “supremum” in the above quantity without changing its value. Thus we see that the above quantity is further bounded by

$$C_2 \sum_{|\alpha|, |\beta| \leq N} \sup_{|y| \leq |x|+1, \text{each } y_k \geq 0} \left| \lim_{j \to \infty} \partial^\beta_y \left( E_{N+2} \ast \rho_j (x-y) \right) \right|.$$ 


$$C_2(|x| + 1)^{|\alpha|} \sum_{|\alpha|, |\beta| \leq N} \sup_{|y| \leq |x|+1, \text{each } y_k \geq 0} \left| \lim_{j \to \infty} \partial^\beta_y \left( E_{N+2} \ast \rho_j (x-y) \right) \right|$$

$$= C_2(|x| + 1)^{|\alpha|} \sum_{|\alpha|, |\beta| \leq N} \sup_{|y| \leq |x|+1, \text{each } y_k \geq 0} \left| \partial^\beta_y E_{N+2} (x-y) \right|$$

$$= C_2(|x| + 1)^{|\alpha|} \sum_{|\alpha|, |\beta| \leq N} \sup_{|y| \leq |x|+1, \text{each } y_k \geq 0} \left| \partial^\beta_y \left( \frac{(x-y)^{(N+1)\bar{1}}}{[(N + 1)\bar{1}]^n} \right) \right|$$

$$= \frac{C_2}{[(N + 1)!]^n} (|x| + 1)^{|\alpha|} \sum_{|\alpha|, |\beta| \leq N, \beta \leq (N+1)\bar{1}} \sup_{|y| \leq |x|+1, \text{each } y_k \geq 0} \left| (x-y)^{(N+1)\bar{1}-\beta} \right|$$

(note the change under the last $\Sigma$ symbol). Since each

$$|(x-y)^{(N+1)\bar{1}-\beta}| \leq (2|x| + 1)^{(N+1)\bar{1}-\beta}$$

for all $y$ in the domain of the above supremum, the last quantity in the previous equation is further bounded by

$$\frac{C_2}{[(N + 1)!]^n} (|x| + 1)^{|\alpha|} \sum_{|\alpha|, |\beta| \leq N, \beta \leq (N+1)\bar{1}} (2|x| + 1)^{(N+1)\bar{1}-\beta}.$$ 

Since this is a bound on $|E_{N+2} \ast u(x)|$, it is clear from here that $E_{N+2} \ast u$ is indeed of polynomial growth. This finally proves the theorem.

∎