Higher Weak Orders of Affine Permutations

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Definition

The **affine symmetric group on** \( n \) **elements** \( \tilde{S}_n \) consists of bijections \( w : \mathbb{Z} \rightarrow \mathbb{Z} \) satisfying:

- \( w(x + n) = w(x) \) for all \( x \in \mathbb{Z} \), and
- \( w(1) + w(2) + \cdots + w(n) = \binom{n+1}{2} \)

The **window notation** of \( w \in \tilde{S}_n \) is \([w(1), \ldots, w(n)]\). E.g., \([-3, 2, 7, 4] \in \tilde{S}_4 \) is the affine permutation:

\[
\begin{array}{cccccccccccc}
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\downarrow \\
\cdots & -7 & -2 & 3 & 0 & -3 & 2 & 7 & 4 & 1 & 6 & 11 & 8 & \cdots \\
\end{array}
\]
Affine Symmetric Group $\tilde{S}_n$

$\tilde{S}_n$ is generated by simple transpositions $s_1, \ldots, s_n$ where

$$s_i = [w(1), \ldots, w(i+1), w(i), \ldots, w(n)], \text{ for } 1 \leq i \leq n-1$$

and

$$s_n = [w(0), w(2), w(3), \ldots, w(n+1)].$$

**Definition**

A reduced word of $w \in \tilde{S}_n$ is a minimal length word $i_1 i_2 \cdots i_\ell$ in the alphabet $[n] := \{1, 2, \ldots, n\}$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$.

A reduced word for $[-3, 2, 7, 4] \in \tilde{S}_4$ is 121343.

How do we know 121343 is minimal length?
Affine Symmetric Group $\tilde{S}_n$

Let $\left(\mathbb{Z}_k\right) := \{(i_1, \ldots, i_k) : i_j \in \mathbb{Z} \text{ and } i_1 < i_2 < \cdots < i_k\}$.

**Definition**

The \textit{n-periodic k-subsets} of $\mathbb{Z}$ is the set $\left(\mathbb{Z}_k\right)_n := \left(\mathbb{Z}_k\right) / \sim_n$ where

$$l \sim_n J \iff l - J = m(n, \ldots, n) \text{ for some } m \in \mathbb{Z}.$$ 

**Definition**

The \textit{2-inversions} of $w \in \tilde{S}_n$ is the set:

$$\text{Inv}_2(w) = \left\{ (i,j) : (i,j) \in \left(\mathbb{Z}_k\right)_n \text{ and } w^{-1}(i) > w^{-1}(j) \right\}.$$  

The \textbf{length} of $w \in \tilde{S}_n$ is $\ell(w) := |\text{Inv}_2(w)|$.

$$\text{Inv}_2([-3, 2, 7, 4]) = \{(1, 2), (1, 3), (2, 3), (1, 4), (1, 7), (4, 7)\}$$
Affine Symmetric Group $\tilde{S}_n$

**Definition**

The **(right) weak order** on $\tilde{S}_n$ is the transitive closure of the relation

$$v \prec w \iff w = vs_i \text{ and } \ell(w) = \ell(v) + 1.$$
Affine Symmetric Group $\tilde{S}_n$

There is a natural injection $\iota : S_n \hookrightarrow \tilde{S}_n$ sending $w \mapsto [w(1), \ldots, w(n)]$. 

\[
\begin{array}{ccc}
[3,2,1] & \quad & [3,1,2] \\
\downarrow & & \downarrow \\
[2,3,1] & \quad & [2,1,3] \\
\quad & s_1 & \quad & \quad & s_2 \\
[3,1,2] & \quad & [1,3,2] \\
\downarrow & & \downarrow \\
[2,1,3] & \quad & [1,2,3] \\
\quad & s_1 & \quad & \quad & s_2 \\
[1,2,3] & \\
\end{array}
\]
Affine Symmetric Group $\tilde{S}_n$

There is a natural injection $\iota : S_n \hookrightarrow \tilde{S}_n$ sending $w \mapsto [w(1), \ldots, w(n)]$. 

![Diagram of the action of $S_n$ on $\tilde{S}_n$]
Affine Symmetric Group $\tilde{S}_n$

There is a natural injection $\iota : S_n \hookrightarrow \tilde{S}_n$ sending $w \mapsto [w(1), \ldots, w(n)]$. 

Diagram:

- $[2,3,1] \xrightarrow{s_1} [3,2,1] \xleftarrow{s_2} [3,1,2]
- [1,2,3] \xrightarrow{s_1} [1,3,2] \xleftarrow{s_2} [2,1,3]
- 121 \xrightarrow{s_1} 212 \xleftarrow{s_2} 121$
Affine Symmetric Group $\tilde{S}_n$

Properties of the weak order:

- Ranked poset with rank function $\ell(w)$.
- In $S_n$, unique min element $\text{Id} = [1, \ldots, n]$ and max element $w_0 = [n, \ldots, 1]$.
- Maximal saturated chains $\leftrightarrow$ reduced words of $w_0$. 
Higher Weak Order \((k = 2)\)

Let \(\mathcal{R}(w)\) be the reduced words for \(w \in \tilde{S}_n\).

**Definition**

Reduced words \(\rho\) and \(\sigma\) differ by a **commutation move** if \(\sigma\) is obtained from \(\rho\) by an adjacent swap \(ij \rightarrow ji\) for \(i \not\equiv j \pm 1 \pmod{n}\). If \(\rho\) and \(\sigma\) differ by a sequence of commutation moves, they are **commutation equivalent**, written \(\rho \sim \sigma\).

**Definition**

Reduced words \(\rho\) and \(\sigma\) differ by a **directed braid move** if \(\sigma\) is obtained from \(\rho\) by an adjacent swap \(i(i+1)i \rightarrow (i+1)i(i+1)\).

The **commutation classes** of \(w\) are \(\mathcal{C}(w) := \mathcal{R}(w)/\sim\).
Higher Weak Order \((k = 2)\)

Let \(G_2(w)\) be a directed graph with vertices \(C(w)\) and directed edges derived from directed braid moves.

E.g. \(w = [3, 2, 1] \in \tilde{S}_3\).
Higher Weak Order ($k = 2$)

Let $G_2(w)$ be a directed graph with vertices $C(w)$ and directed edges derived from directed braid moves.

E.g. $w = [-3, 2, 7, 4] \in \tilde{S}_4$.

![Diagram](image_url)

$[212434] = [214234]$

$343 \rightarrow 434$

$121 \rightarrow 212$

$[212343]$

$[121434]$

$121 \rightarrow 212$

$343 \rightarrow 434$

$[121343] = [123143]$
Higher Weak Order ($k = 2$)

**Theorem (Matsumoto (1964), Tits (1969))**

For any $w \in \tilde{S}_n$ and $\rho, \sigma \in R(w)$, there is a sequence of commutation and braid moves that takes $\rho$ to $\sigma$.

In fact, $G_2(w)$ looks like it could be a ranked poset!

**Theorem (Manin, Schechtman (1989))**

Let $w_0 = [n, n - 1, \ldots, 1]$. Then $G_2(w_0)$ is the Hasse diagram of a ranked poset with a unique minimal and maximal element.

This ranked poset is the second higher weak order $B_{n,2}(w_0)$. What is the rank function?
Higher Weak Order ($k = 2$)

Reduced words $\rho \in \mathcal{R}(w)$ biject to linear orders on the 2-inversion set $\text{Inv}_2(w)$.

E.g. for $[-3, 2, 7, 4] \in \tilde{S}_4$, we have $121343 \leftrightarrow 12 < 13 < 23 < 14 < 17 < 47$.

\[
\begin{array}{ccccccccccccc}
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\end{array}
\]

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\begin{array}{ccccccccccccc}
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```
... -3  -2  -1  0  1  2  3  4  5  6  7  8  ...
```

```
... -2  -3  -1  0  2  1  3  4  6  5  7  8  ...
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\downarrow \\
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\downarrow \\
\ldots & -1 & -2 & 0 & -3 & 3 & 2 & 4 & 1 & 7 & 6 & 8 & 5 & \ldots \\
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\cdots & -7 & -2 & 0 & 3 & -3 & 2 & 4 & 7 & 1 & 6 & 8 & 11 & \cdots
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\]
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\end{array}
\]
Higher Weak Order \((k = 2)\)

**Definition**

The set of \(k\)-inversions of \(w \in \tilde{S}_n\) is

\[
\text{Inv}_k(w) := \left\{ (i_1, \ldots, i_k) \in \binom{\mathbb{Z}}{k}_n : w^{-1}(i_1) > \cdots > w^{-1}(i_k) \right\}.
\]

**Definition**

Let \(\rho \in \mathcal{R}(w)\) with associated linear order \((i_1, j_1) <_\rho (i_2, j_2) <_\rho \cdots <_\rho (i_\ell, j_\ell)\). The **reversal set** of \(\rho\) is

\[
\text{Rev}(\rho) := \{(i, j, k) \in \text{Inv}_3(w) : (j, k) <_\rho (i, k) <_\rho (i, k)\}.
\]

Commutation classes are uniquely defined by their reversal set. The rank function on \(B_{n,2}(w_0)\) is \(|\text{Rev}(\rho)|\).
Higher Weak Order \((k = 2)\)

\[
[212434] = [214234]
\]

\[
343 \rightarrow 434 \quad 121 \rightarrow 212
\]

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[212343] \quad [121434]
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[121343] = [123143]
\]
Higher Weak Order \((k = 2)\)
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Higher Weak Order ($k = 2$)

123 < 147

\{123\} \rightarrow \{123, 147\}

\{147\} \leftarrow 147 < 123
Manin-Schechtman Higher Weak Orders

Classically only defined for the longest permutation $w_0 = [n, n - 1, \ldots, 1]$ which has inversion set $\text{Inv}_k(w_0) = \binom{[n]}{k}$.

For $X \in \binom{\mathbb{Z}}{k}_n$, the packet of $X = (x_1, \ldots, x_k)$ is

$$P(X) := \{X_1, X_2, \ldots, X_k\},$$

where $X_i = (x_1, \ldots, \hat{x}_i, \ldots, x_k)$. The lexicographic (lex) order on $P(X)$ is $X_k < \cdots < X_1$ and the antilexicographic (antilex) order is $X_1 < \cdots < X_k$.

**Definition**

A linear order $<_\rho$ on $\text{Inv}_k(w_0)$ is **admissible** if $<_\rho$ restricts to either the lex or antilex order on any packet $P(X)$ for $X \in \text{Inv}_{k+1}(w_0)$. The set of admissible orders on $\text{Inv}_k(w_0)$ is denoted $A_{n,k}(w_0)$. 
Manin-Schechtman Higher Weak Orders

Some linear orders for $n = 5$, $k = 3$:

- 123 < 124 < 125 < 134 < 135 < 145 < 234 < 235 < 245 < 345
- 123 < 124 < 125 < 134 < 145 < 135 < 234 < 235 < 245 < 345
- 123 < 124 < 125 < 134 < 135 < 145 < 345 < 245 < 235 < 234

**Definition**

Two admissible orders $<_{\rho}$ and $<_{\sigma}$ on $\text{Inv}_k(w_0)$ are **commutation equivalent** if $<_{\sigma}$ differs from $<_{\rho}$ by a sequence of adjacent swaps $X < Y \rightarrow Y < X$ where $X, Y$ do not lie in any common packet $P(Z)$.

Denote the set of $k$-admissible orders modulo commutation equivalence by $B_{n,k}(w_0) := A_{n,k}(w_0)/\sim$. 
Manin-Schechtman Higher Weak Orders

**Definition**

The set of **reversals** of an admissible order \( \prec_\rho \) on \( \text{Inv}_k(w_0) \) is

\[
\text{Rev}(\prec_\rho) = \{ X \in \text{Inv}_{k+1}(w_0) : \prec_\rho \text{ is the antilex order on } P(X) \}.
\]

**Definition**

Two admissible orders \( \prec_\rho \) and \( \prec_\sigma \) on \( \text{Inv}_k(w_0) \) differ by a **directed packet flip** if \( \prec_\sigma \) is obtained from \( \prec_\rho \) by reversing the order on a lex packet \( P(X) \) that forms a chain in \( \prec_\rho \):

\[
X_{k+1} \prec_\rho \cdots \prec_\rho X_1 \rightarrow X_1 \prec_\sigma \cdots \prec_\sigma X_{k+1}.
\]
Manin-Schechtman Higher Weak Orders

<table>
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<th>$k = 1$</th>
<th>$k = 2$</th>
<th>General $k$</th>
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<td>admissible orders</td>
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<tr>
<td>simple tranposition</td>
<td>directed braid move</td>
<td>directed packet flip</td>
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Let $G_k(w_0)$ be the directed graph with vertex set $B_{n,k}(w_0)$ and edges $[<\rho] \to [<\sigma]$ if some $<\rho' \in [<\rho]$ and $<\sigma' \in [<\sigma]$ differ by a directed packet flip.

**Theorem (Manin, Schechtman (1989))**

For $1 \leq k \leq n$, the following hold:

- Elements of $B_{n,k}(w_0)$ are uniquely determined by their reversal set.
- The directed graph $G_{n,k}(w_0)$ is the Hasse diagram of a partial order $\leq$ on $B_{n,k}(w_0)$, equivalent to single step inclusion of reversal sets.
- The poset $(B_{n,k}(w_0), \leq)$ is a ranked poset with unique min and max elements whose reversal sets are $\emptyset$ and $\text{Inv}_{k+1}(w_0)$ respectively. The rank function is $|\text{Rev}([<\rho])|$. 
- For $2 \leq k \leq n$, elements of $A_{n,k}(w_0)$ are in bijection with maximal chains in $B_{n,k-1}(w_0)$.

The $k$th higher weak order of $w_0$ is $(B_{n,k}(w_0), \leq)$.
Manin-Schechtman Higher Weak Orders

Other equivalent characterizations: consistent sets, oriented matroid extensions, and zonotopal tilings.
Manin-Schechtman Higher Weak Orders

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**Definition**

A set $U \subseteq \binom{[n]}{k}$ is **consistent** if its intersection $U \cap P(X)$ for all $X \in \binom{[n]}{k+1}$ is either a prefix or suffix of $P(X)$ under lex order.

**Theorem (Ziegler (1993))**

There is a bijection between $(B_{n,k}(w_0), \leq)$ and consistent subsets of $\binom{[n]}{k+1}$ ordered by single step inclusion.
Manin-Schechtman Higher Weak Orders

Other equivalent characterizations: consistent sets, oriented matroid extensions, and zonotopal tilings.

\[^n]_k\) bijects with vertices in cyclic arrangement \(X_{c,n-n-k}\). e.g. \(n = 5, k = 3\).
Manin-Schechtman Higher Weak Orders

Other equivalent characterizations: consistent sets, oriented matroid extensions, and zonotopal tilings.

\binom{n}{k} bijects with vertices in cyclic arrangement \( X_{c}^{n,n-k} \). e.g. \( n = 5, k = 3 \).
Manin-Schechtman Higher Weak Orders

Other equivalent characterizations: consistent sets, oriented matroid extensions, and zonotopal tilings.

**Theorem (Ziegler (1993))**

There is a bijection between \((B_{n,k}(w_0), \leq)\) and the poset of uniform single element extensions of the affine alternating oriented matroid \(C^{n,n-k}\).
Manin-Schechtman Higher Weak Orders

Other equivalent characterizations: consistent sets, oriented matroid extensions, and zonotopal tilings.

**Theorem (Thomas (1993))**

*There is a bijection between $(B_{n,k}(w_0), \leq)$ and subsets $U$ of the $k$-faces of $[0, 1]^n$ such that $T(U)$ tiles $T([0, 1]^n)$ for some totally positive map $T : \mathbb{R}^n \to \mathbb{R}^k$.***
Manin-Schechtman Higher Weak Orders

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$B_{n,2}(w)$ bijects with rhombic tilings of the symmetric $2n$-gon. E.g. for $n = 4$, [123121] corresponds to

![Diagram of rhombic tiling]

1 \quad 2 \quad 3

1 \quad 2 \quad 1

1 \quad 1
Higher Weak Orders in $\tilde{S}_n$

**Question:** Starting from the weak order $(\tilde{S}_n, \leq)$, what if instead of taking maximal chains from $\text{Id}$ to $w_0$ we took maximal chains up to an arbitrary $w$?

We need a generalized notion of admissible orders.

**Definition**

A linear order $<_\rho$ on $\text{Inv}_k(w)$ is **admissible** if it satisfies the following properties:

- For $X \in \text{Inv}_{k+1}(w)$, $<_\rho$ restricts to the lex or antilex order on $P(X)$.
- For $X, Y \in \binom{\mathbb{Z}}{k}_n$ such that $X \leq_P Y$, we have $X <_\rho Y$. 
Higher Weak Orders in $\tilde{S}_n$

**Lemma**

For $w \in \tilde{S}_n$, $k > 1$, and $X \in \left(\mathbb{Z}\right)_n$, we have $X \in \text{Inv}_k(w)$ if and only if $P(X) \subseteq \text{Inv}_{k-1}(w)$.

**Lemma**

Let $w \in \tilde{S}_n$ and $X = [x_1, \ldots, x_k] \in \left(\mathbb{Z}\right)_n$. The intersection $P(X) \cap \text{Inv}_{k-1}(w)$ is one of the following:

- the empty set $\emptyset$, or
- a singleton set $\{X_i\}$, or
- a consecutive pair $\{X_i, X_{i+1}\}$ for some $1 \leq i \leq k-1$, or
- all of $P(X)$. 
Higher Weak Orders in $\tilde{S}_n$

**Definition**

Two $n$-periodic $k$-sets $X = [x_1, \ldots, x_k]$ and $Y = [y_1, \ldots, y_k]$ are **congruent modulo** $n$, denoted $X \equiv Y \pmod{n}$ if $x_i \equiv y_i \pmod{n}$ for all $1 \leq i \leq k$.

**Lemma**

If $X = [x_1, \ldots, x_k], Y = [y_1, \ldots, y_k] \in \text{Inv}_k(w)$ such that

$$\{x_1 \pmod{n}, \ldots, x_k \pmod{n}\} = \{y_1 \pmod{n}, \ldots, y_k \pmod{n}\}$$

as sets, then $X \equiv Y \pmod{n}$.
Higher Weak Orders in $\tilde{S}_n$

Let $v_i^{(k)} := (0, \ldots, 0, 1, \ldots, 1)$ with $i$ zeroes followed by $k - i$ ones.

**Definition**

The **permanent poset** $(\text{Inv}_k(w), \leq_P)$ is the transitive closure of the relation defined by the following.

- For $X \in \binom{\mathbb{Z}}{k+1}_n$ with $P(X) \cap \text{Inv}_k(w) = \{X_i, X_{i+1}\}$, we have $X_{i+1} \leq_P X_i$ if $k - i$ even and $X_i \leq_P X_{i+1}$ if $k - i$ odd.

- For $X, Y \in \binom{\mathbb{Z}}{k+1}_n$ with $Y = X + v_i^{(k)}$, we have $Y \leq_P X$ if $k - i$ is even and $X \leq_P Y$ if $k - i$ is odd.
Higher Weak Orders in $\tilde{S}_n$

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- For $X, Y \in \binom{\mathbb{Z}}{k+1}^n$ with $Y = X + v_i^{(k)}$, we have $X \leq_P Y$ if $k - i$ is odd and $Y \leq_P X$ if $k - i$ is even.

E.g. $n = 3, k = 2, i = 1, w = [3, 1, 2], X = (1, 2, 3)$:

$$\text{Inv}_2(w) = \{23, 13\} = \{X_1, X_2\}.$$  

$$B_{n,k}(w) = \{23 < 13\} = \{X_1 < X_2\}.$$
Higher Weak Orders in $\tilde{S}_n$

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E.g. $n = 3$, $k = 2$, $i = 2$, $w = [2, 3, 1]$, $X = (1, 2, 3)$:

$$\text{Inv}_2(w) = \{13, 12\} = \{X_2, X_3\}.$$  

$$B_{n,k}(w) = \{12 < 13\} = \{X_3 < X_2\}.$$
Higher Weak Orders in $\tilde{S}_n$

**Definition**

The **permanent poset** $(\text{Inv}_k(w), \leq_P)$ is the transitive closure of the relation defined by the following.

- For $X \in \left(\mathbb{Z}_{k+1}\right)^n$ with $P(X) \cap \text{Inv}_k(w) = \{X_i, X_{i+1}\}$, we have $X_i \leq_P X_{i+1}$ if $k - i$ is odd and $X_{i+1} \leq_P X_i$ if $k - i$ is even.

- For $X, Y \in \left(\mathbb{Z}_{k+1}\right)^n$ with $Y = X + \nu_i^{(k)}$, we have $X \leq_P Y$ if $k - i$ is odd and $Y \leq_P X$ if $k - i$ is even.

E.g. $n = 3$, $k = 3$, $i = 1$, $w = [6, 2, -2]$:

$$\text{Inv}_2(w) = \{12, 13, 23, 15, 16, 26\} \quad \text{Inv}_3(w) = \{123, 126, 156, 159\}.$$
### Higher Weak Orders in $\tilde{S}_n$

#### Definition

A linear order $\langle \rho \rangle$ on $\text{Inv}_k(w)$ is **admissible** if it satisfies the following properties:

- For $X \in \text{Inv}_{k+1}(w)$, $\langle \rho \rangle$ restricts to the lex or antilex order on $P(X)$.
- For $X, Y \in \binom{\mathbb{Z}}{k}_n$ such that $X \leq_P Y$, we have $X <_\rho Y$.

#### Definition

The set of **reversals** of an admissible order $\langle \rho \rangle$ on $\text{Inv}_k(w)$ is

$$\text{Rev}(\langle \rho \rangle) = \{ X \in \text{Inv}_{k+1}(w) : \langle \rho \rangle \text{ is the antilex order on } P(X) \}.$$ 

The set of admissible orders on $\text{Inv}_k(w)$ is denoted $A_{n,k}(w)$. The set of reversal sets of $A_{n,k}(w)$ is denoted $B_{n,k}(w)$. 
Higher Weak Orders in \( \tilde{S}_n \)

**Theorem (Billey-Elias-Liu-C.)**

For all \( w \in \tilde{S}_n \), \( B_{n,2}(w) \) is a ranked poset under single step inclusion of reversal sets, with unique min element \( \emptyset \) and unique max element \( \text{Inv}_3(w) \). Elements of \( B_{n,2}(w) \) biject to maximal chains in \([\text{Id}, w]\) in the weak order.

What about general \( k \)? We conjectured and verified computationally for \( n \leq 6 \) and \( \ell(w) \leq 15 \).

**Conjecture**

For all \( w \in \tilde{S}_n \) and \( 2 \leq k \leq n \), \( B_{n,k}(w) \) is a ranked poset under single step inclusion of reversal sets with unique min element \( \emptyset \) and unique max element \( \text{Inv}_{k+1}(w) \). Maximal chains of \( B_{n,k}(w) \) biject with admissible orders in \( A_{n,k+1}(w) \).
Weaving Patterns

We can visualize reduced words via **wiring diagrams**.

E.g. the wiring diagram of $121343 \in \mathcal{R}([-3, 2, 7, 4])$ is:

![Wiring Diagram]

**Definition**

For $w \in \tilde{S}_n$, the **weaving pattern** associated to a reduced word $\rho \in \mathcal{R}(w)$ is a function $P_\rho : [n] \to \{-1, +1\}^*$ such that $P_\rho(i)$ is the sequence of up (+1) crossings and down (-1) crossings of the wire labeled $i$. 
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\[
\begin{array}{ccccccc}
S_1 & S_2 & S_1 & S_3 & S_4 & S_3 \\
4 & 3 & 2 & 1 & 4 & 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
4 & 7 & 2 & -3 & - & ++ \\
3 & - & - & -- & - & ++ \\
2 & - & + & + & - & ++ \\
1 & + & + & + & + & + \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
Weaving Patterns

**Definition**
The length of a weaving pattern $P_\rho$ for $\rho \in \mathcal{R}(w)$ is $\ell(P_\rho) := \ell(w)$.

**Definition**
The content of a weaving pattern $P_\rho$ is a pair of sequences $(a_i)$ and $(b_i)$ where $a_i$ is the number of +1’s in $P_\rho(i)$ and $b_i$ is the number of -1’s in $P_\rho(i)$.

**Lemma**
Let $w \in \tilde{S}_n$, $\rho \in \mathcal{R}(w)$, and $(a_i), (b_i)$ be the content of $P_\rho$. Then the following hold:

- For all $i \in [n]$, $w^{-1}(i) = i + a_i - b_i$
- $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = \ell(P_\rho)$
- If $\sigma \in \mathcal{R}(w)$ and $\sigma \sim \rho$ then $P_\sigma = P_\rho$.

**Open Problem**: Given an arbitrary function $P : [n] \to \{-1, 1\}^*$, decide if $P = P_\rho$ for some reduced word $\rho$. 
Lemma

Let $w, w' \in \tilde{S}_n$, $\rho \in \mathcal{R}(w)$, and $\sigma \in \mathcal{R}(w')$. Then $P_\rho = P_\sigma$ if and only if $w = w'$ and $\rho \sim \sigma$.

Weaving patterns biject with commutation classes. What do directed braid moves do?
Weaving Patterns

Lemma

Let \( w, w' \in \tilde{S}_n \), \( \rho \in R(w) \), and \( \sigma \in R(w') \). Then \( P_\rho = P_\sigma \) if and only if \( w = w' \) and \( \rho \sim \sigma \).

Weaving patterns biject with commutation classes. What do directed braid moves do?
Lemma

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**Lemma**

Let $w, w' \in \tilde{S}_n$, $\rho \in \mathcal{R}(w)$, and $\sigma \in \mathcal{R}(w')$. Then $P_\rho = P_\sigma$ if and only if $w = w'$ and $\rho \sim \sigma$.

Weaving patterns biject with commutation classes. What do directed braid moves do?

**Lemma**

If $\rho, \sigma \in \mathcal{R}(w)$ and $\sigma$ differs from $\rho$ by a directed braid move, then $P_\sigma$ differs from $P_\rho$ by a single adjacent swap $-+ \rightarrow +-$. 

Weaving Patterns

**Theorem (Billey-Elias-Liu-C.)**

Let \( w \in \tilde{S}_n, \rho \in \mathcal{R}(w), \) and suppose that \( P_{\rho}(i) \) contains a contiguous subword \(-+\) for some \( i \in [n] \). Then there exists \( \sigma \in \mathcal{R}(w) \) such that \([\rho]\) and \([\sigma]\) differ by a directed braid move of the form \( j(j+1)j \rightarrow (j+1)j(j+1) \) for some \( j \in [n] \).

**Corollary**

For arbitrary \( w \in \tilde{S}_n, B_{n,2}(w) \) has a unique min element \([\rho]\) and a unique max element \([\sigma]\) such that \( P_{\rho} \) has no contiguous subword \(+−\) and \( P_{\sigma} \) has no contiguous subword \(−+\).
Theorem (Billey-Elias-Liu-C.)

Let $w \in \tilde{S}_n$, $\rho \in \mathcal{R}(w)$, and suppose that $P_{\rho}(i)$ contains a contiguous subword $-+ \, \text{for some } i \in [n]$. Then there exists $\sigma \in \mathcal{R}(w)$ such that $[\rho]$ and $[\sigma]$ differ by a directed braid move of the form $j(j+1)j \to (j+1)j(j+1)$ for some $j \in [n]$.

Proof.

WLOG, assume that the $-+$ crossings occur in row 2. Then there is a contiguous subword $1\tau 1$ where $\tau \in \{2, 3, 4, \ldots, n-1\}^*$. If $\tau$ contains only one 2, then done. Otherwise, write $\tau$ contains a subword $2\tau'2$ with $\tau' \in \{3, 4, \ldots, n-1\}^*$ and induct.
Weaving Patterns

**Theorem (Billey-Elias-Liu-C.)**

For all $w \in \tilde{S}_n$, $B_{n,2}(w)$ is a ranked poset under single step inclusion of reversal sets, with unique min element $\emptyset$ and unique max element $\text{Inv}_3(w)$. Elements of $B_{n,2}(w)$ biject to maximal chains in $[\text{Id}, w]$ in the weak order.
Enumeration of $B_{n,k}(w)$

**Theorem (Stanley (1984))**

The cardinality of $\mathcal{R}(w_0)$ for $w_0 \in S_n$ is equal to the number of standard Young tableaux of shape $(n-1, n-2, \ldots, 1)$.

What about $B_{n,2}(w) = C(w_0)$?

**Theorem (Knuth (1992))**

The cardinality of $C(w_0)$ for $w_0 \in S_n$ is asymptotically equal to $2^{\Theta(n^2)}$.

For comparison, $|\mathcal{R}(w_0)|$ is asymptotically equal to $2^{\Theta(n^2 \log n)}$. 
Enumeration of $B_{n,k}(w)$

**Theorem (Ziegler (1993))**

For all $n \geq 4$, we have

- $|B_{n,n}(w_0)| = 1$,
- $|B_{n,n-1}(w_0)| = 2$,
- $|B_{n,n-2}(w_0)| = 2n$, and
- $|B_{n,n-3}(w_0)| = 2^n + n2^{n-2} - 2n$.

**Corollary (Billey-Elias-Liu-C.)**

Let $w \in \tilde{S}_n$, $\rho \in \mathcal{R}(w)$ and $(a_i), (b_i)$ be the content of $P_\rho$. Then

$$|B_{n,2}(w)| \leq \prod_{i=1}^{n} \binom{a_i + b_i}{a_i}.$$
Future Work

- Generalize $B_{n,k}(w)$ to infinite biclosed sets (Barkley-Speyer) and infinite reduced words (Lam-Pylyavskyy).
- Generalize weaving patterns to encode elements of $B_{n,k}(w_0)$ for $k > 2$.
- Find a simple criterion that characterizes weaving patterns.
- Asymptotics of $|B_{n,k}(w_0)|$. 