Higher Weak Orders of Affine Permutations

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Affine Symmetric Group $\widetilde{S}_n$

Definition

The **affine symmetric group on n elements** $\widetilde{S}_n$ consists of bijections $w : \mathbb{Z} \to \mathbb{Z}$ satisfying:

- $w(x+n) = w(x)$ for all $x \in \mathbb{Z}$, and
- $w(1) + w(2) + \cdots + w(n) = \binom{n+1}{2}$

The **window notation** of $w \in \widetilde{S}_n$ is $[w(1), \ldots, w(n)]$. E.g., $[-3, 2, 7, 4] \in \widetilde{S}_4$ is the affine permutation:

$$\begin{array}{ccccccccccccccc} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ & & & & & & & \downarrow & & & & & & \\ \cdots & -7 & -2 & 3 & 0 & -3 & 2 & 7 & 4 & 1 & 6 & 11 & 8 & \cdots \end{array}$$

Affine Symmetric Group $\widetilde{S}_n$

$\widetilde{S}_n$ is generated by **simple transpositions** $s_1, \ldots, s_n$ where

$$s_i = [w(1), \ldots, w(i+1), w(i), \ldots, w(n)], \text{ for } 1 \leq i \leq n-1$$

and

$$s_n = [w(0), w(2), w(3), \ldots, w(n+1)].$$

Definition

A **reduced word** of $w \in \widetilde{S}_n$ is a minimal length word $i_1 i_2 \cdots i_\ell$ in the alphabet $[n] := \{1, 2, \ldots, n\}$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$.

A reduced word for $[-3, 2, 7, 4] \in \widetilde{S}_4$ is 121343.

How do we know 121343 is minimal length?

Affine Symmetric Group $\widetilde{S}_n$

Let $\binom{\mathbb{Z}}{k} := \{(i_1, \ldots, i_k) \ : \ i_j \in \mathbb{Z} \text{ and } i_1 < i_2 < \cdots < i_k\}$.

Definition

The *n*-**periodic** *k*-**subsets** of $\mathbb{Z}$ is the set $\binom{\mathbb{Z}}{k}_n := \binom{\mathbb{Z}}{k} / \sim_n$ where

$$I \sim_n J \Leftrightarrow I - J = m(n, \ldots, n) \text{ for some } m \in \mathbb{Z}.$$

Definition

The **2-inversions** of $w \in \widetilde{S}_n$ is the set:

$$\mathrm{Inv}_2(w) = \left\{(i,j) \ : \ (i,j) \in \binom{\mathbb{Z}}{k}_n \text{ and } w^{-1}(i) > w^{-1}(j)\right\}.$$

The **length** of $w \in \widetilde{S}_n$ is $\ell(w) := |\,\mathrm{Inv}_2(w)|$.

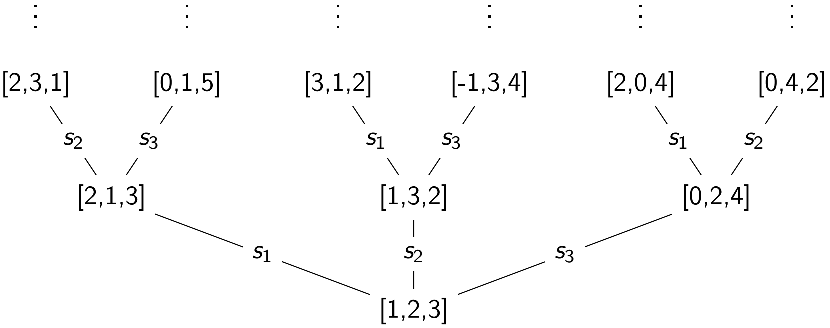
$$\mathrm{Inv}_2([-3,2,7,4]) = \{(1,2),(1,3),(2,3),(1,4),(1,7),(4,7)\}$$

Affine Symmetric Group $\widetilde{S}_n$

Definition

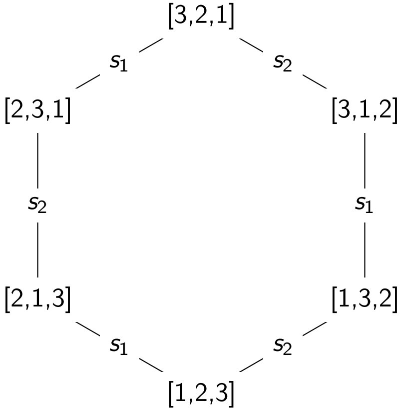
The **(right) weak order** on $\widetilde{S}_n$ is the transitive closure of the relation

$$v \lessdot w \Leftrightarrow w = vs_i \text{ and } \ell(w) = \ell(v) + 1.$$



Affine Symmetric Group $\widetilde{S}_n$

There is a natural injection $\iota : S_n \hookrightarrow \widetilde{S}_n$ sending $w \mapsto [w(1), \ldots, w(n)]$.



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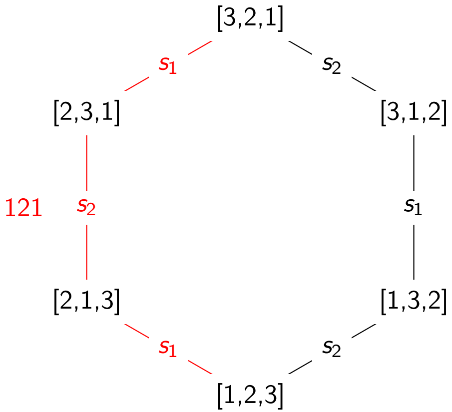

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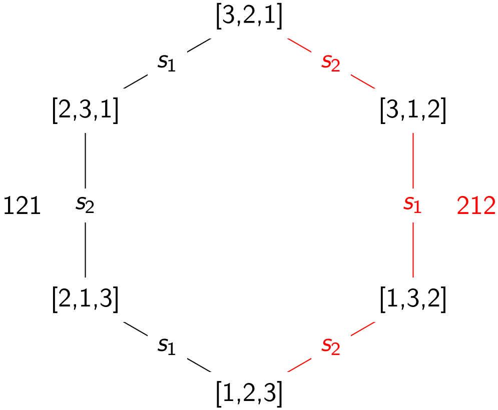


Affine Symmetric Group $\widetilde{S}_n$

Properties of the weak order:

- Ranked poset with rank function $\ell(w)$.
- In S_n, unique min element $\mathrm{Id} = [1, \ldots, n]$ and max element $w_0 = [n, \ldots, 1]$.
- Maximal saturated chains $\leftrightarrow$ reduced words of w_0.

Higher Weak Order ($k = 2$)

Let $\mathcal{R}(w)$ be the reduced words for $w \in \widetilde{S}_n$.

Definition

Reduced words ρ and σ differ by a **commutation move** if σ is obtained from ρ by an adjacent swap $ij \to ji$ for $i \not\equiv j \pm 1 \pmod{n}$. If ρ and σ differ by a sequence of commutation moves, they are **commutation equivalent**, written $\rho \sim \sigma$.

Definition

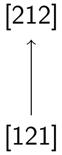
Reduced words ρ and σ differ by a **directed braid move** if σ is obtained from ρ by an adjacent swap $i(i+1)i \to (i+1)i(i+1)$.

The **commutation classes** of w are $\mathcal{C}(w) := \mathcal{R}(w)/\sim$.

Higher Weak Order ($k = 2$)

Let $G_2(w)$ be a directed graph with vertices $\mathcal{C}(w)$ and directed edges derived from directed braid moves.

E.g. $w = [3, 2, 1] \in \widetilde{S}_3$.



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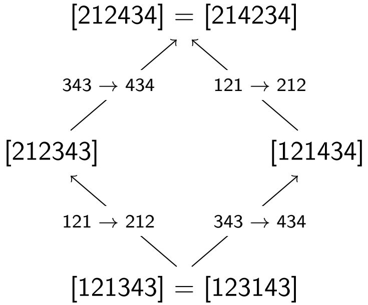

Higher Weak Order ($k = 2$)

Theorem (Matsumoto (1964), Tits (1969))

For any $w \in \widetilde{S}_n$ and $\rho, \sigma \in \mathcal{R}(w)$, there is a sequence of commutation and braid moves that takes ρ to σ.

In fact, $G_2(w)$ looks like it could be a ranked poset!

Theorem (Manin, Schechtman (1989))

Let $w_0 = [n, n-1, \ldots, 1]$. Then $G_2(w_0)$ is the Hasse diagram of a ranked poset with a unique minimal and maximal element.

This ranked poset is the second higher weak order $B_{n,2}(w_0)$. What is the rank function?

Higher Weak Order ($k = 2$)

Reduced words $\rho \in \mathcal{R}(w)$ biject to linear orders on the 2-inversion set $\mathrm{Inv}_2(w)$.

E.g. for $[-3, 2, 7, 4] \in \widetilde{S}_4$, we have $121343 \leftrightarrow 12 < 13 < 23 < 14 < 17 < 47$.

$$\cdots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \cdots$$

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E.g. for $[-3, 2, 7, 4] \in \widetilde{S}_4$, we have $12134\textcolor{red}{3} \leftrightarrow 12 < 13 < 23 < 14 < 17 < \textcolor{red}{47}$.

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Higher Weak Order ($k = 2$)

Definition

The set of k-**inversions** of $w \in \widetilde{S}_n$ is

$$\mathrm{Inv}_k(w) := \left\{ (i_1, \ldots, i_k) \in \binom{\mathbb{Z}}{k}_n \ : \ w^{-1}(i_1) > \cdots > w^{-1}(i_k) \right\}.$$

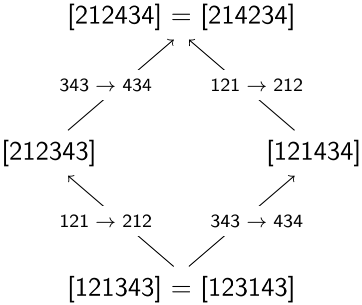
Definition

Let $\rho \in \mathcal{R}(w)$ with associated linear order $(i_1, j_1) <_\rho (i_2, j_2) <_\rho \cdots <_\rho (i_\ell, j_\ell)$. The **reversal set** of ρ is

$$\mathrm{Rev}(\rho) := \{(i,j,k) \in \mathrm{Inv}_3(w) \ : \ (j,k) <_\rho (i,k) <_\rho (i,k)\}.$$

Commutation classes are uniquely defined by their reversal set. The rank function on $B_{n,2}(w_0)$ is $|\,\mathrm{Rev}(\rho)|$.

Higher Weak Order ($k = 2$)



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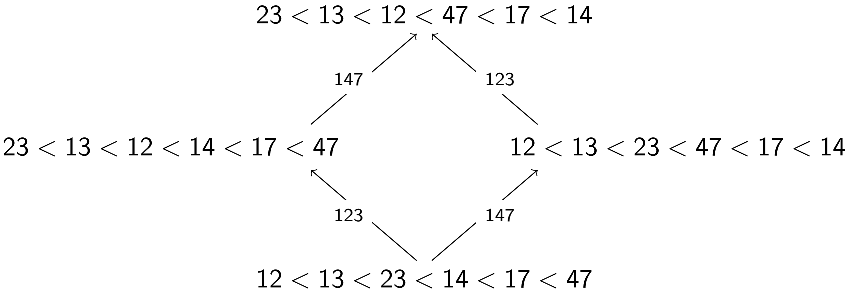

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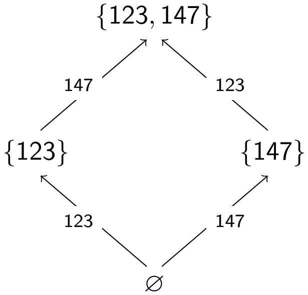
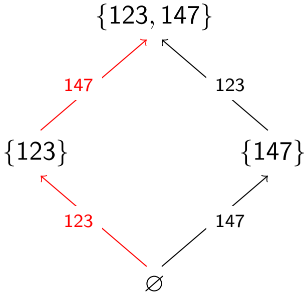


Higher Weak Order ($k = 2$)

$123 < 147$



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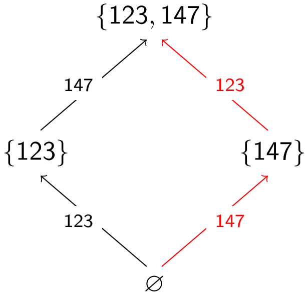

$123 < 147$

$147 < 123$

Manin-Schechtman Higher Weak Orders

Classically only defined for the longest permutation $w_0 = [n, n-1, \ldots, 1]$ which has inversion set $\mathrm{Inv}_k(w_0) = \binom{[n]}{k}$.

For $X \in \binom{\mathbb{Z}}{k}_n$, the **packet** of $X = (x_1, \ldots, x_k)$ is

$$P(X) \coloneqq \{X_1, X_2, \ldots, X_k\},$$

where $X_i = (x_1, \ldots, \widehat{x_i}, \ldots, x_k)$. The **lexicographic (lex) order** on $P(X)$ is $X_k < \cdots < X_1$ and the **antilexicographic (antilex) order** is $X_1 < \cdots < X_k$.

Definition

A linear order $<_\rho$ on $\mathrm{Inv}_k(w_0)$ is **admissible** if $<_\rho$ restricts to either the lex or antilex order on any packet $P(X)$ for $X \in \mathrm{Inv}_{k+1}(w_0)$. The set of admissible orders on $\mathrm{Inv}_k(w_0)$ is denoted $A_{n,k}(w_0)$.

Manin-Schechtman Higher Weak Orders

Some linear orders for $n = 5, k = 3$:

✓ $123 < 124 < 125 < 134 < 135 < 145 < 234 < 235 < 245 < 345$

✗ $123 < 124 < 125 < 134 < 145 < 135 < 234 < 235 < 245 < 345$

✓ $123 < 124 < 125 < 134 < 135 < 145 < 345 < 245 < 235 < 234$

Definition

Two admissible orders $<_\rho$ and $<_\sigma$ on $\mathrm{Inv}_k(w_0)$ are **commutation equivalent** if $<_\sigma$ differs from $<_\rho$ by a sequence of adjacent swaps $X < Y \to Y < X$ where X, Y do not lie in any common packet $P(Z)$.

Denote the set of k-admissible orders modulo commutation equivalence by $B_{n,k}(w_0) := A_{n,k}(w_0)/\sim$.

Manin-Schechtman Higher Weak Orders

Definition

The set of **reversals** of an admissible order $<_\rho$ on $\mathrm{Inv}_k(w_0)$ is

$$\mathrm{Rev}(<_\rho) = \{X \in \mathrm{Inv}_{k+1}(w_0) \ : <_\rho \text{ is the antilex order on } P(X)\}.$$

Definition

Two admissible orders $<_\rho$ and $<_\sigma$ on $\mathrm{Inv}_k(w_0)$ differ by a **directed packet flip** if $<_\sigma$ is obtained from $<_\rho$ by reversing the order on a lex packet $P(X)$ that forms a chain in $<_\rho$:

$$X_{k+1} \lessdot_\rho \cdots \lessdot_\rho X_1 \to X_1 \lessdot_\sigma \cdots \lessdot_\sigma X_{k+1}.$$

Manin-Schechtman Higher Weak Orders

$k = 1$	$k = 2$	general k
permutations	reduced words	admissible orders
inversions	reversal sets	reversal sets
simple tranposition	directed braid move	directed packet flip

Manin-Schechtman Higher Weak Orders

Let $G_k(w_0)$ be the directed graph with vertex set $B_{n,k}(w_0)$ and edges $[<_\rho] \to [<_\sigma]$ if some $<_{\rho'} \in [<_\rho]$ and $<_{\sigma'} \in [<_\sigma]$ differ by a directed packet flip.

Theorem (Manin, Schechtman (1989))

For $1 \leq k \leq n$, the following hold:

- *Elements of $B_{n,k}(w_0)$ are uniquely determined by their reversal set.*
- *The directed graph $G_{n,k}(w_0)$ is the Hasse diagram of a partial order $\leq$ on $B_{n,k}(w_0)$, equivalent to single step inclusion of reversal sets.*
- *The poset $(B_{n,k}(w_0), \leq)$ is a ranked poset with unique min and max elements whose reversal sets are $\varnothing$ and $\mathrm{Inv}_{k+1}(w_0)$ respectively. The rank function is $|\mathrm{Rev}([<_\rho])|$.*
- *For $2 \leq k \leq n$, elements of $A_{n,k}(w_0)$ are in bijection with maximal chains in $B_{n,k-1}(w_0)$.*

The k**th higher weak order** of w_0 is $(B_{n,k}(w_0), \leq)$.

Manin-Schechtman Higher Weak Orders

Other equivalent characterizations: consistent sets, oriented matroid extensions, and zonotopal tilings.

Manin-Schechtman Higher Weak Orders

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Definition

A set $U \subseteq \binom{[n]}{k}$ is **consistent** if its intersection $U \cap P(X)$ for all $X \in \binom{[n]}{k+1}$ is either a prefix or suffix of $P(X)$ under lex order.

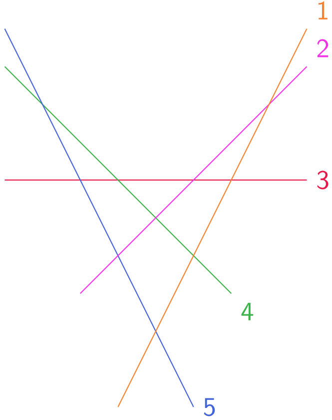
Theorem (Ziegler (1993))

There is a bijection between $(B_{n,k}(w_0), \leq)$ and consistent subsets of $\binom{[n]}{k+1}$ ordered by single step inclusion.

Manin-Schechtman Higher Weak Orders

Other equivalent characterizations: consistent sets, oriented matroid extensions, and zonotopal tilings.

$\binom{[n]}{k}$ bijects with vertices in cyclic arrangement $X_c^{n,n-k}$. e.g. $n = 5$, $k = 3$.



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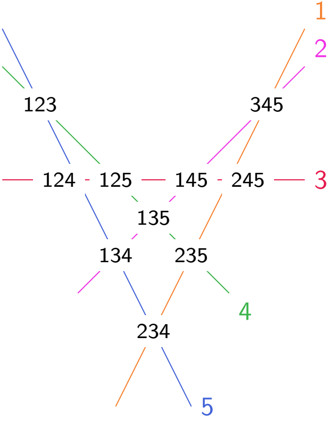

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Theorem (Ziegler (1993))

There is a bijection between $(B_{n,k}(w_0), \leq)$ *and the poset of uniform single element extensions of the affine alternating oriented matroid* $C^{n,n-k}$.

Manin-Schechtman Higher Weak Orders

Other equivalent characterizations: consistent sets, oriented matroid extensions, and zonotopal tilings.

Theorem (Thomas (1993))

There is a bijection between $(B_{n,k}(w_0), \leq)$ *and subsets* U *of the* k*-faces of* $[0,1]^n$ *such that* $T(U)$ *tiles* $T([0,1]^n)$ *for some totally positive map* $T : \mathbb{R}^n \to \mathbb{R}^k$.

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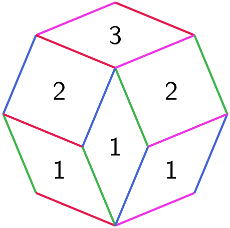
$B_{n,2}(w)$ bijects with rhombic tilings of the symmetric $2n$-gon. E.g. for $n = 4$, [123121] corresponds to



Higher Weak Orders in $\widetilde{S}_n$

Question: Starting from the weak order $(\widetilde{S}_n, \leq)$, what if instead of taking maximal chains from Id to w_0 we took maximal chains up to an arbitrary w?

We need a generalized notion of admissible orders.

Definition

A linear order $<_\rho$ on $\mathrm{Inv}_k(w)$ is **admissible** if it satisfies the following properties:

- For $X \in \mathrm{Inv}_{k+1}(w)$, $<_\rho$ restricts to the lex or antilex order on $P(X)$.
- For $X, Y \in \binom{\mathbb{Z}}{k}_n$ such that $X \leq_P Y$, we have $X <_\rho Y$.

Higher Weak Orders in $\widetilde{S}_n$

Lemma

For $w \in \widetilde{S}_n$, $k > 1$, and $X \in \binom{\mathbb{Z}}{k}_n$, we have $X \in \mathrm{Inv}_k(w)$ if and only if $P(X) \subseteq \mathrm{Inv}_{k-1}(w)$.

Lemma

Let $w \in \widetilde{S}_n$ and $X = [x_1, \ldots, x_k] \in \binom{\mathbb{Z}}{k}_n$. The intersection $P(X) \cap \mathrm{Inv}_{k-1}(w)$ is one of the following:

- *the empty set $\varnothing$, or*
- *a singleton set $\{X_i\}$, or*
- *a consecutive pair $\{X_i, X_{i+1}\}$ for some $1 \leq i \leq k-1$, or*
- *all of $P(X)$.*

Higher Weak Orders in $\widetilde{S}_n$

Definition

Two n-periodic k-sets $X = [x_1, \ldots, x_k]$ and $Y = [y_1, \ldots, y_k]$ are **congruent modulo** n, denoted $X \equiv Y \pmod{n}$ if $x_i \equiv y_i \pmod{n}$ for all $1 \leq i \leq k$.

Lemma

If $X = [x_1, \ldots, x_k], Y = [y_1, \ldots, y_k] \in \mathrm{Inv}_k(w)$ *such that*

$$\{x_1 \ (mod\ n), \ldots, x_k \ (mod\ n)\} = \{y_1 \ (mod\ n), \ldots, y_k \ (mod\ n)\}$$

as sets, then $X \equiv Y \pmod{n}$.

Higher Weak Orders in $\widetilde{S}_n$

Let $v_i^{(k)} := (0, \dots, 0, 1, \dots, 1)$ with i zeroes followed by $k - i$ ones.

Definition

The **permanent poset** $(\mathrm{Inv}_k(w), \leq_P)$ is the transitive closure of the relation defined by the following.

- For $X \in \binom{\mathbb{Z}}{k+1}_n$ with $P(X) \cap \mathrm{Inv}_k(w) = \{X_i, X_{i+1}\}$, we have $X_{i+1} \leq_P X_i$ if $k - i$ even and $X_i \leq_P X_{i+1}$ if $k - i$ odd.
- For $X, Y \in \binom{\mathbb{Z}}{k+1}_n$ with $Y = X + v_i^{(k)}$, we have $Y \leq_P X$ if $k - i$ is even and $X \leq_P Y$ if $k - i$ is odd.

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- For $X, Y \in \binom{\mathbb{Z}}{k+1}_n$ with $Y = X + v_i^{(k)}$, we have $X \leq_P Y$ if $k - i$ is odd and $Y \leq_P X$ if $k - i$ is even.

E.g. $n = 3$, $k = 2$, $i = 1$, $w = [3, 1, 2]$, $X = (1, 2, 3)$:

$$\mathrm{Inv}_2(w) = \{23, 13\} = \{X_1, X_2\}.$$

$$B_{n,k}(w) = \{23 < 13\} = \{X_1 < X_2\}.$$

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- For $X \in \binom{\mathbb{Z}}{k+1}_n$ with $P(X) \cap \mathrm{Inv}_k(w) = \{X_i, X_{i+1}\}$, we have $X_i \leq_P X_{i+1}$ if $k-i$ is odd and $X_{i+1} \leq_P X_i$ if $k-i$ is even.
- For $X, Y \in \binom{\mathbb{Z}}{k+1}_n$ with $Y = X + v_i^{(k)}$, we have $X \leq_P Y$ if $k-i$ is odd and $Y \leq_P X$ if $k-i$ is even.

E.g. $n=3$, $k=2$, $i=2$, $w=[2,3,1]$, $X=(1,2,3)$:

$$\mathrm{Inv}_2(w) = \{13, 12\} = \{X_2, X_3\}.$$

$$B_{n,k}(w) = \{12 < 13\} = \{X_3 < X_2\}.$$

Higher Weak Orders in $\widetilde{S}_n$

Definition

The **permanent poset** $(\mathrm{Inv}_k(w), \leq_P)$ is the transitive closure of the relation defined by the following.

- For $X \in \binom{\mathbb{Z}}{k+1}_n$ with $P(X) \cap \mathrm{Inv}_k(w) = \{X_i, X_{i+1}\}$, we have $X_i \leq_P X_{i+1}$ if $k - i$ is odd and $X_{i+1} \leq_P X_i$ if $k - i$ is even.
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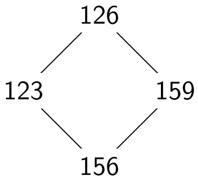
E.g. $n = 3$, $k = 3$, $i = 1$, $w = [6, 2, -2]$:

$$\mathrm{Inv}_2(w) = \{12, 13, 23, 15, 16, 26\} \quad \mathrm{Inv}_3(w) = \{123, 126, 156, 159\}.$$



Higher Weak Orders in $\widetilde{S}_n$

Definition

A linear order $<_\rho$ on $\mathrm{Inv}_k(w)$ is **admissible** if it satisfies the following properties:

- For $X \in \mathrm{Inv}_{k+1}(w)$, $<_\rho$ restricts to the lex or antilex order on $P(X)$.
- For $X, Y \in \binom{\mathbb{Z}}{k}_n$ such that $X \leq_P Y$, we have $X <_\rho Y$.

Definition

The set of **reversals** of an admissible order $<_\rho$ on $\mathrm{Inv}_k(w)$ is

$$\mathrm{Rev}(<_\rho) = \{X \in \mathrm{Inv}_{k+1}(w) \ : <_\rho \text{ is the antilex order on } P(X)\}.$$

The set of admissible orders on $\mathrm{Inv}_k(w)$ is denoted $A_{n,k}(w)$. The set of reversal sets of $A_{n,k}(w)$ is denoted $B_{n,k}(w)$.

Higher Weak Orders in $\widetilde{S}_n$

Theorem (Billey-Elias-Liu-C.)

For all $w \in \widetilde{S}_n$, $B_{n,2}(w)$ is a ranked poset under single step inclusion of reversal sets, with unique min element $\varnothing$ and unique max element $\mathrm{Inv}_3(w)$. Elements of $B_{n,2}(w)$ biject to maximal chains in $[\mathrm{Id}, w]$ in the weak order.

What about general k? We conjectured and verified computationally for $n \leq 6$ and $\ell(w) \leq 15$.

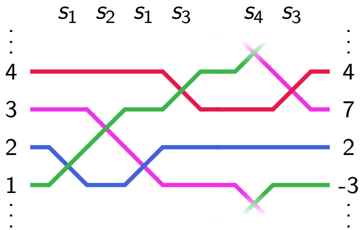
Conjecture

For all $w \in \widetilde{S}_n$ and $2 \leq k \leq n$, $B_{n,k}(w)$ is a ranked poset under single step inclusion of reversal sets with unique min element $\varnothing$ and unique max element $\mathrm{Inv}_{k+1}(w)$. Maximal chains of $B_{n,k}(w)$ biject with admissible orders in $A_{n,k+1}(w)$.

Weaving Patterns

We can visualize reduced words via **wiring diagrams**.

E.g. the wiring diagram of $121343 \in \mathcal{R}([-3, 2, 7, 4])$ is:



Definition

For $w \in \widetilde{S}_n$, the **weaving pattern** associated to a reduced word $\rho \in \mathcal{R}(w)$ is a function $P_\rho : [n] \to \{-1, +1\}^*$ such that $P_\rho(i)$ is the sequence of up $(+1)$ crossings and down (-1) crossings of the wire labeled i.

Weaving Patterns

Definition

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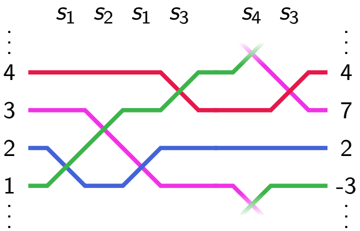


4	$-+$
3	$----$
2	$-+$
1	$++++$

Weaving Patterns

Definition

The **length** of a weaving pattern P_ρ for $\rho \in \mathcal{R}(w)$ is $\ell(P_\rho) := \ell(w)$.

Definition

The **content** of a weaving pattern P_ρ is a pair of sequences (a_i) and (b_i) where a_i is the number of $+1$'s in $P_\rho(i)$ and b_i is the number of -1's in $P_\rho(i)$.

Lemma

Let $w \in \widetilde{S}_n$, $\rho \in \mathcal{R}(w)$, and $(a_i), (b_i)$ be the content of P_ρ. Then the following hold:

- *For all $i \in [n]$, $w^{-1}(i) = i + a_i - b_i$*
- $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = \ell(P_\rho)$
- *If $\sigma \in \mathcal{R}(w)$ and $\sigma \sim \rho$ then $P_\sigma = P_\rho$.*

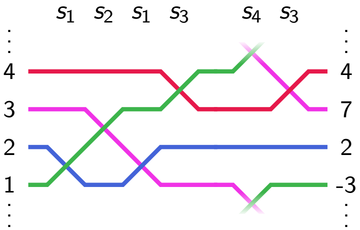
Open Problem: Given an arbitrary function $P : [n] \to \{-1, 1\}^*$, decide if $P = P_\rho$ for some reduced word ρ.

Weaving Patterns

Lemma

Let $w, w' \in \widetilde{S}_n$, $\rho \in \mathcal{R}(w)$, and $\sigma \in \mathcal{R}(w')$. Then $P_\rho = P_\sigma$ if and only if $w = w'$ and $\rho \sim \sigma$.

Weaving patterns biject with commutation classes. What do directed braid moves do?



4	$-+$
3	$- - --$
2	$-+$
1	$+ + ++$

Weaving Patterns

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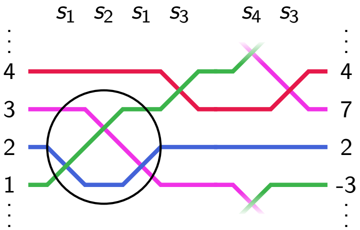

4	$-+$
3	$--\ -\ -$
2	$-+$
1	$++\ +\ +$

Weaving Patterns

Lemma

Let $w, w' \in \widetilde{S}_n$, $\rho \in \mathcal{R}(w)$, and $\sigma \in \mathcal{R}(w')$. Then $P_\rho = P_\sigma$ if and only if $w = w'$ and $\rho \sim \sigma$.

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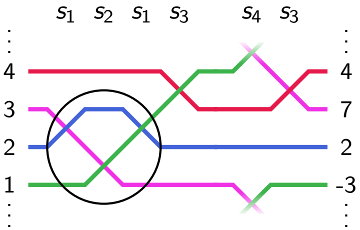


4	$-+$
3	$--\ -\ -$
2	$+-$
1	$++\ +\ +$

Weaving Patterns

Lemma

Let $w, w' \in \widetilde{S}_n$, $\rho \in \mathcal{R}(w)$, and $\sigma \in \mathcal{R}(w')$. Then $P_\rho = P_\sigma$ if and only if $w = w'$ and $\rho \sim \sigma$.

Weaving patterns biject with commutation classes. What do directed braid moves do?

Lemma

If $\rho, \sigma \in \mathcal{R}(w)$ and σ differs from ρ by a directed braid move, then P_σ differs from P_ρ by a single adjacent swap $-+ \to +-$.

Weaving Patterns

Theorem (Billey-Elias-Liu-C.)

Let $w \in \widetilde{S}_n$, $\rho \in \mathcal{R}(w)$, and suppose that $P_\rho(i)$ contains a contiguous subword $-+$ for some $i \in [n]$. Then there exists $\sigma \in \mathcal{R}(w)$ such that $[\rho]$ and $[\sigma]$ differ by a directed braid move of the form $j(j+1)j \to (j+1)j(j+1)$ for some $j \in [n]$.

Corollary

For arbitrary $w \in \widetilde{S}_n$, $B_{n,2}(w)$ has a unique min element $[\rho]$ and a unique max element $[\sigma]$ such that P_ρ has no contiguous subword $+-$ and P_σ has no contiguous subword $-+$.

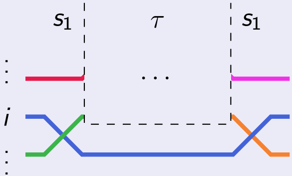
Weaving Patterns

Theorem (Billey-Elias-Liu-C.)

Let $w \in \widetilde{S}_n$, $\rho \in \mathcal{R}(w)$, and suppose that $P_\rho(i)$ contains a contiguous subword $-+$ for some $i \in [n]$. Then there exists $\sigma \in \mathcal{R}(w)$ such that $[\rho]$ and $[\sigma]$ differ by a directed braid move of the form $j(j+1)j \to (j+1)j(j+1)$ for some $j \in [n]$.

Proof.

WLOG, assume that the $-+$ crossings occur in row 2. Then there is a contiguous subword $1\tau 1$ where $\tau \in \{2, 3, 4, \ldots, n-1\}^*$.



If τ contains only one 2, then done. Otherwise, write τ contains a subword $2\tau' 2$ with $\tau' \in \{3, 4, \ldots, n-1\}^*$ and induct. $\square$

Weaving Patterns

Theorem (Billey-Elias-Liu-C.)

For all $w \in \widetilde{S}_n$, $B_{n,2}(w)$ is a ranked poset under single step inclusion of reversal sets, with unique min element $\varnothing$ and unique max element $\mathrm{Inv}_3(w)$. Elements of $B_{n,2}(w)$ biject to maximal chains in $[\mathrm{Id}, w]$ in the weak order.

Enumeration of $B_{n,k}(w)$

Theorem (Stanley (1984))

The cardinality of $\mathcal{R}(w_0)$ for $w_0 \in S_n$ is equal to the number of standard Young tableaux of shape $(n-1, n-2, \ldots, 1)$.

What about $B_{n,2}(w) = \mathcal{C}(w_0)$?

Theorem (Knuth (1992))

The cardinality of $\mathcal{C}(w_0)$ for $w_0 \in S_n$ is asymptotically equal to $2^{\Theta(n^2)}$.

For comparison, $|\mathcal{R}(w_0)|$ is asymptotically equal to $2^{\Theta(n^2 \log n)}$.

Enumeration of $B_{n,k}(w)$

Theorem (Ziegler (1993))

For all $n \geq 4$, we have

- $|B_{n,n}(w_0)| = 1$,
- $|B_{n,n-1}(w_0)| = 2$,
- $|B_{n,n-2}(w_0)| = 2n$, *and*
- $|B_{n,n-3}(w_0)| = 2^n + n2^{n-2} - 2n$.

Corollary (Billey-Elias-Liu-C.)

Let $w \in \widetilde{S}_n$, $\rho \in \mathcal{R}(w)$ and $(a_i),(b_i)$ be the content of P_ρ. Then

$$|B_{n,2}(w)| \leq \prod_{i=1}^{n} \binom{a_i + b_i}{a_i}.$$

Future Work

- Generalize $B_{n,k}(w)$ to infinite biclosed sets (Barkley-Speyer) and infinite reduced words (Lam-Pylyavskyy).
- Generalize weaving patterns to encode elements of $B_{n,k}(w_0)$ for $k > 2$.
- Find a simple criterion that characterizes weaving patterns.
- Asymptotics of $|B_{n,k}(w_0)|$.