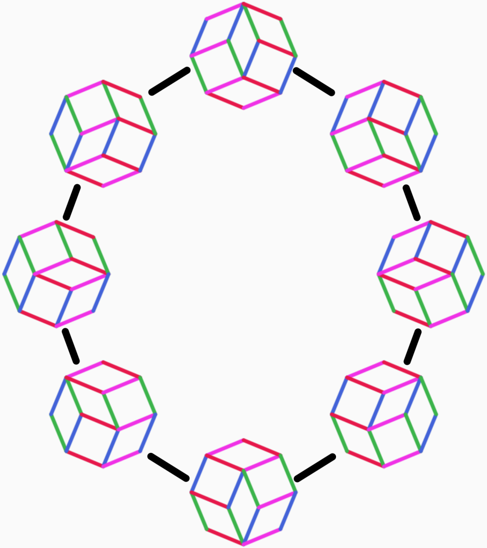
Commutation Classes of Permutations

Herman Chau

University of Washington

Commutation Classes of [4321]



What is a Commutation Class?

Permutations

$S_n :=$ symmetric group on n elements

Permutations

S_n := symmetric group on n elements

σ_i := adjacent transposition swapping i and $i+1$

Permutations

$S_n :=$ symmetric group on n elements

$\sigma_i :=$ adjacent transposition swapping i and $i+1$

The longest permutation w_0 sends $i \mapsto n - i + 1$.

Permutations

S_n := symmetric group on n elements

σ_i := adjacent transposition swapping i and $i+1$

The longest permutation w_0 sends $i \mapsto n-i+1$.

Example ($n = 4$)

$$\begin{aligned} w_0 &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix} \\ &= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1. \end{aligned}$$

Permutations

S_n := symmetric group on n elements

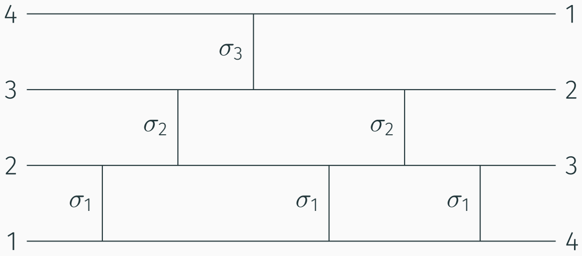
σ_i := adjacent transposition swapping i and $i+1$

The longest permutation w_0 sends $i \mapsto n-i+1$.

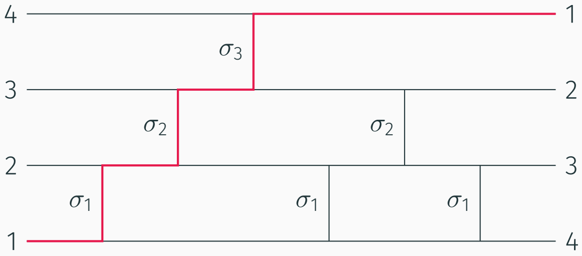
Example ($n = 4$)

$$\begin{aligned} w_0 &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix} \\ &= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1. \end{aligned}$$

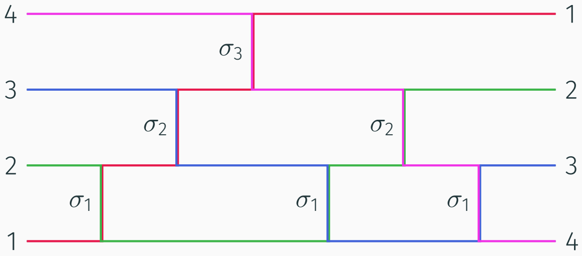
Wiring Diagrams



Wiring Diagrams



Wiring Diagrams



Reduced Expressions

Definition

A reduced expression for $w \in S_n$ is a product of adjacent transpositions $w = \sigma_{i_1} \cdots \sigma_{i_j}$ of minimum length. The set of all reduced expressions for w is denoted $\mathcal{R}(w)$.

Reduced Expressions

Definition

A reduced expression for $w \in S_n$ is a product of adjacent transpositions $w = \sigma_{i_1} \cdots \sigma_{i_j}$ of minimum length. The set of all reduced expressions for w is denoted $\mathcal{R}(w)$.

Definition

The length of a permutation w, denoted $\ell(w)$, is the length of any reduced expression for w.

Reduced Expressions

Definition

A reduced expression for $w \in S_n$ is a product of adjacent transpositions $w = \sigma_{i_1} \cdots \sigma_{i_j}$ of minimum length. The set of all reduced expressions for w is denoted $\mathcal{R}(w)$.

Definition

The length of a permutation w, denoted $\ell(w)$, is the length of any reduced expression for w.

Fact

The longest permutation $w_0 \in S_n$ has length $\binom{n}{2}$.

Commutation and Braid Relations

Given a reduced expression of $w \in S_n$, we can obtain new reduced expressions via **commutation** and **braid** relations.

- **Commutation:** Let $1 \leq i < i+1 < j \leq n-1$. Then $\sigma_i \sigma_j = \sigma_j \sigma_i$.
- **Braid:** Let $1 \leq i \leq n-2$. Then $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Commutation and Braid Relations

Given a reduced expression of $w \in S_n$, we can obtain new reduced expressions via **commutation** and **braid** relations.

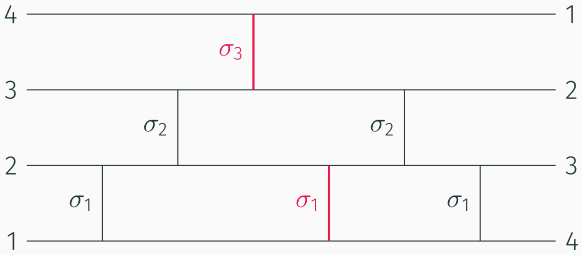
- **Commutation:** Let $1 \leq i < i+1 < j \leq n-1$. Then $\sigma_i \sigma_j = \sigma_j \sigma_i$.
- **Braid:** Let $1 \leq i \leq n-2$. Then $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Example (n=4)

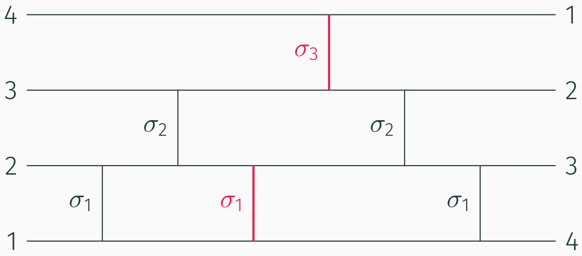
$$\sigma_1 \sigma_2 \underline{\boldsymbol{\sigma_3 \sigma_1}} \sigma_2 \sigma_1 \overset{\text{commute}}{=} \sigma_1 \sigma_2 \underline{\boldsymbol{\sigma_1 \sigma_3}} \sigma_2 \sigma_1$$

$$\sigma_1 \sigma_2 \sigma_3 \underline{\boldsymbol{\sigma_1 \sigma_2 \sigma_1}} \overset{\text{braid}}{=} \sigma_1 \sigma_2 \sigma_3 \underline{\boldsymbol{\sigma_2 \sigma_1 \sigma_2}}$$

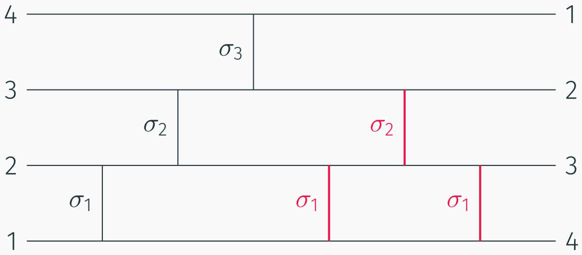
Commutation and Braid Relations



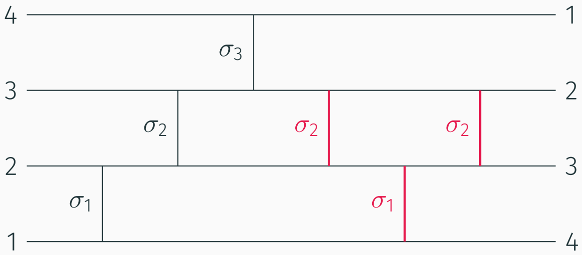
$\downarrow$ commute $\sigma_3\sigma_1$



Commutation and Braid Relations



↓ braid $\sigma_1\sigma_2\sigma_1$



Word Property

Theorem (Matsumoto, Tits)

Any two reduced expressions in $\mathcal{R}(w)$ are connected by a sequence of commutation and braid relations.

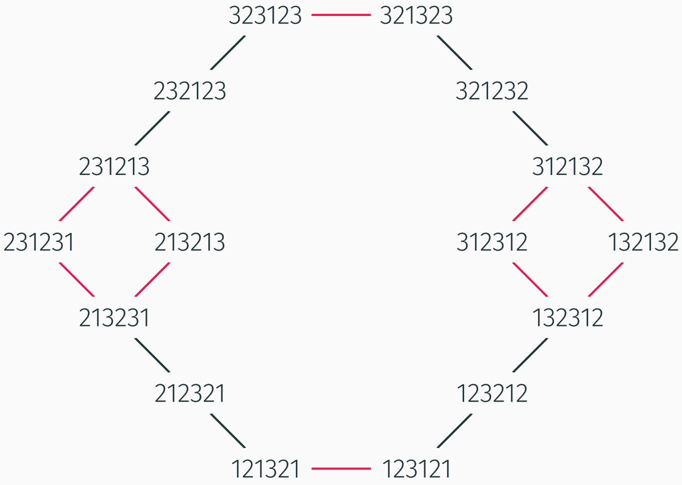
Word Property

Theorem (Matsumoto, Tits)

Any two reduced expressions in $\mathcal{R}(w)$ are connected by a sequence of commutation and braid relations.

We can visualize this as a graph G whose vertex set is $\mathcal{R}(w)$ and (ρ_1, ρ_2) is an edge if and only if ρ_1, ρ_2 differ by a commutation or braid relation.

Reduced Expression Graph for $[4, 3, 2, 1]$



Commutation Classes

Definition

Two reduced expressions ρ_1, ρ_2 are commutation equivalent, denoted $\rho_1 \sim \rho_2$ if they differ by a series of commutation relations.

Commutation Classes

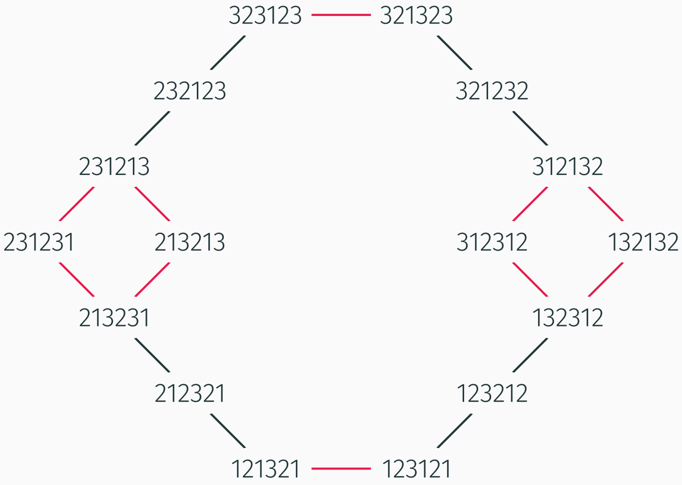
Definition

Two reduced expressions ρ_1, ρ_2 are commutation equivalent, denoted $\rho_1 \sim \rho_2$ if they differ by a series of commutation relations.

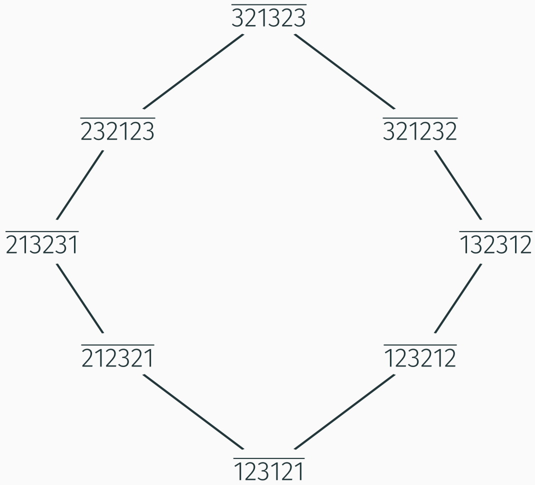
Definition

$\mathcal{C}(w) := \mathcal{R}(w)/\sim$ is the set of commutation classes of w.

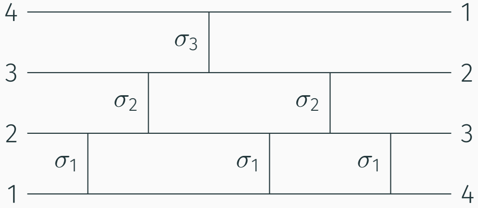
Reduced Expression Graph for $[4, 3, 2, 1]$

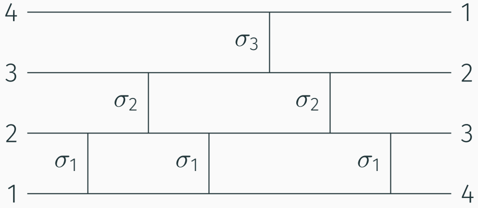


Commutation Class Graph for $[4, 3, 2, 1]$

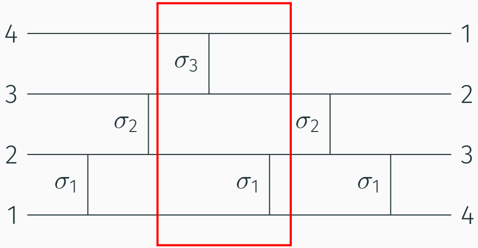


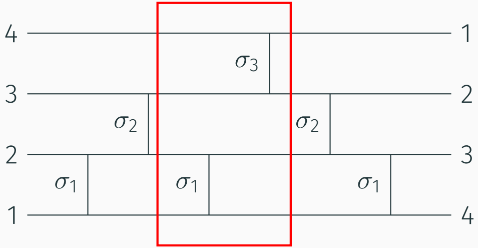
Wiring Diagrams of Commutation Classes



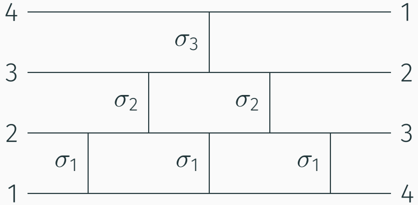


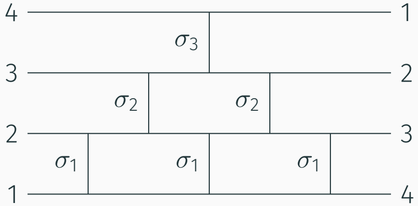
Wiring Diagrams of Commutation Classes





Wiring Diagrams of Commutation Classes



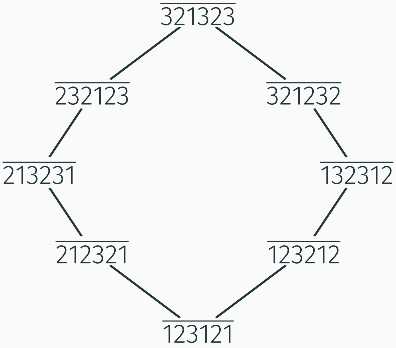


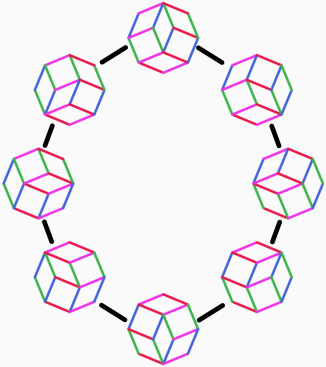
Elnitsky's Bijection

Theorem (Elnitsky, 1993)

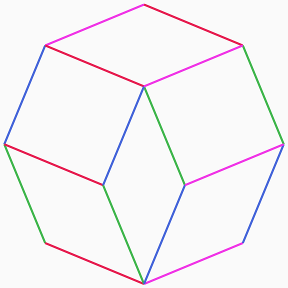
Let $n \geq 2$. There is a bijection between $\mathcal{C}(w_0)$ and rhombic tilings of the regular $2n$-gon.

Bijection in S_4

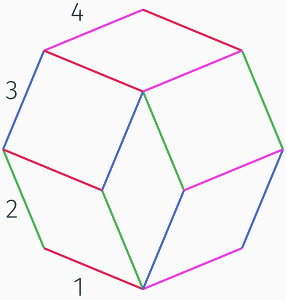




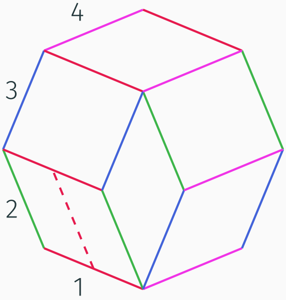
Proof Without Words



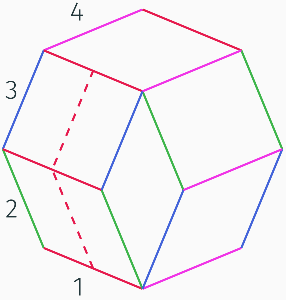
Proof Without Words



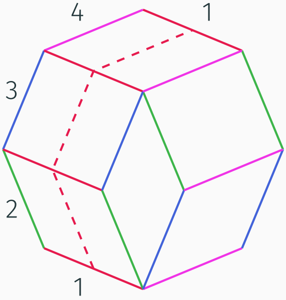
Proof Without Words



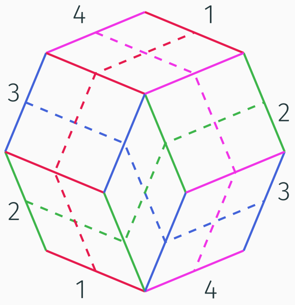
Proof Without Words



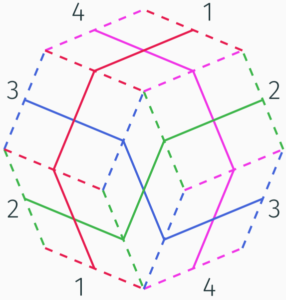
Proof Without Words



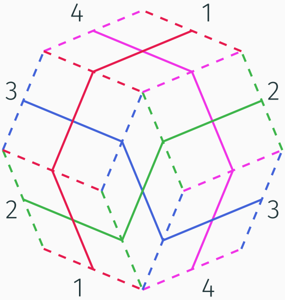
Proof Without Words

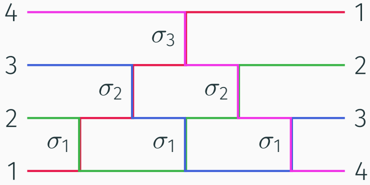


Proof Without Words



Proof Without Words





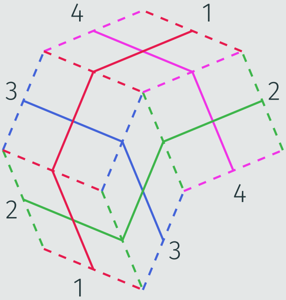
Bijection for General w

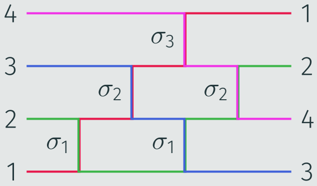
For general $w \in S_n$, $\mathcal{C}(w)$ is in bijection with rhombic tilings of a specific shape, not necessarily the regular $2n$-gon.

Bijection for General w

For general $w \in S_n$, $\mathcal{C}(w)$ is in bijection with rhombic tilings of a specific shape, not necessarily the regular $2n$-gon.

Example ($w = [3, 4, 2, 1]$)





Counting Commutation Classes

Counting Reduced Expressions of w_0

Counting Reduced Expressions of w_0

Theorem (Stanley, 1984) (Edelman and Greene, 1987)

The number of reduced expressions $|\mathcal{R}(w_0)|$ in S_n is equal to the number of standard Young tableaux of shape $(n-1, n-2, \ldots, 1)$. By the hook-length formula, this is equal to $\binom{n}{2}!/1^{n-1}3^{n-2}\cdots(2n-3)^1$.

Counting Reduced Expressions of w_0

Theorem (Stanley, 1984) (Edelman and Greene, 1987)

The number of reduced expressions $|\mathcal{R}(w_0)|$ in S_n is equal to the number of standard Young tableaux of shape $(n-1, n-2, \ldots, 1)$. By the hook-length formula, this is equal to $\binom{n}{2}!/1^{n-1}3^{n-2}\cdots(2n-3)^1$.

n	$\lvert\mathcal{R}(w_0)\rvert$
1	1
2	1
3	2
4	16
5	768
6	292864
...	...

Counting Reduced Expressions of w_0

Theorem (Stanley, 1984) (Edelman and Greene, 1987)

The number of reduced expressions $|\mathcal{R}(w_0)|$ in S_n is equal to the number of standard Young tableaux of shape $(n-1, n-2, \dots, 1)$. By the hook-length formula, this is equal to $\binom{n}{2}!/1^{n-1}3^{n-2}\cdots(2n-3)^1$.

n	$\|\mathcal{R}(w_0)\|$
1	1
2	1
3	2
4	16
5	768
6	292864
...	...

Asymptotically, $|\mathcal{R}(w_0)| = 2^{\Theta(n^2 \log n)}$.

Counting Commutation Classes of w_0

No exact formula is known for the number of commutation classes $|\mathcal{C}(w_0)|$.

Counting Commutation Classes of w_0

No exact formula is known for the number of commutation classes $|\mathcal{C}(w_0)|$.

n	$\lvert\mathcal{C}(w_0)\rvert$
1	1
2	1
3	2
4	8
5	62
6	908
...	...

Counting Commutation Classes of w_0

No exact formula is known for the number of commutation classes $|\mathcal{C}(w_0)|$.

n	$\lvert\mathcal{C}(w_0)\rvert$
1	1
2	1
3	2
4	8
5	$62 = 2 \times 31$
6	$908 = 2^2 \times 227$
...	...

Asymptotics of $|\mathcal{C}(w_0)|$

Theorem (Knuth, 1992)

The number of commutation classes of the longest word $w_0 \in S_n$ grows as $2^{\Theta(n^2)}$.

Asymptotics of $|\mathcal{C}(w_0)|$

Theorem (Knuth, 1992)

The number of commutation classes of the longest word $w_0 \in S_n$ grows as $2^{\Theta(n^2)}$.

What is the constant c in the exponent?

- (Knuth, 1992) $\sim 0.1667 \leq c \leq \sim 0.7924$
- (Felsner, 1997) $c \leq \sim 0.6974$
- (Felsner and Valtr, 2011) $\sim 0.1887 \leq c \leq\sim 0.6571$
- (Dumitrescu and Mandal, 2020) $\sim 0.2083 \leq c$

Asymptotics of $|\mathcal{C}(w_0)|$

Theorem (Knuth, 1992)

The number of commutation classes of the longest word $w_0 \in S_n$ grows as $2^{\Theta(n^2)}$.

What is the constant c in the exponent?

- (Knuth, 1992) $\sim 0.1667 \leq c \leq \sim 0.7924$
- **(Felsner, 1997)** $c \leq \sim 0.6974$
- **(Felsner and Valtr, 2011)** $\sim 0.1887 \leq c \leq \sim 0.6571$
- (Dumitrescu and Mandal, 2020) $\sim 0.2083 \leq c$

Upper Bound on $|\mathcal{C}(w_0)|$

Definition

Let $\overline{\rho} \in \mathcal{C}(w_0)$. The weaving pattern of ρ is a function $P_{\overline{\rho}} : [n] \to \{+, -\}^*$ that records the sequence of up $(+)$ and down $(-)$ crossings in the wiring diagram of $\overline{\rho}$.

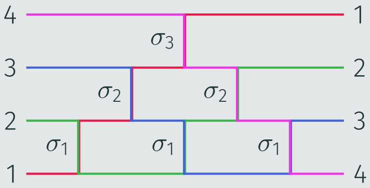
Upper Bound on $|\mathcal{C}(w_0)|$

Definition

Let $\overline{\rho} \in \mathcal{C}(w_0)$. The weaving pattern of ρ is a function $P_{\overline{\rho}} : [n] \to \{+,-\}^*$ that records the sequence of up $(+)$ and down $(-)$ crossings in the wiring diagram of $\overline{\rho}$.

Example

4	$- - -$
3	$- - +$
2	$- + +$
1	$+ + +$



Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Proof.

It suffices to find an inverse map that takes a weaving pattern $P_{\overline{\rho}}$ back to a commutation class $\overline{\rho}$.

1. Find i and $i+1$ such that $P_{\overline{\rho}}(i)$ begins with a $+$ and $P_{\overline{\rho}}(i+1)$ begins with a $-$.
2. Record σ_i, delete the leading $+$ and $-$, and swap rows i and $i+1$ in the weaving pattern.
3. Repeat until the weaving pattern is empty.

□

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{c|c} 4 & - - - \\ 3 & - - + \\ 2 & - + + \\ 1 & + + + \end{array}$$

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{c|ccc} 4 & - & - & - \\ 3 & - & - & + \\ 2 & - & + & + \\ 1 & + & + & + \end{array}$$

Rows 1 and 2 have a leading $+$ and $-$ respectively.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{c|l} 4 & - - - \\ 3 & - - + \\ 2 & ++ \\ 1 & ++ \end{array} \qquad \sigma_1$$

Delete leading signs, swap, and record σ_1.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{c|l} 4 & - - - \\ 3 & - - + \\ 2 & + + \\ 1 & + + \end{array} \qquad \sigma_1$$

Rows 2 and 3 have a leading $+$ and $-$ respectively.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{r|l} 4 & - - - \\ 3 & + \\ 2 & -+ \\ 1 & ++ \end{array} \qquad \sigma_1\sigma_2$$

Delete leading signs, swap, and record σ_2.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{c|l} 4 & -\,-\,- \\ 3 & + \\ 2 & -+ \\ 1 & ++ \end{array} \qquad \sigma_1\sigma_2$$

Choice of either rows 1 and 2, or rows 3 and 4.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{c|l} 4 & - - - \\ 3 & + \\ 2 & -+ \\ 1 & ++ \end{array} \qquad \sigma_1\sigma_2$$

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{c|c} 4 & \\ 3 & -- \\ 2 & -+ \\ 1 & ++ \end{array} \qquad \sigma_1\sigma_2\sigma_3$$

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\begin{array}{c|l} 4 & \\ 3 & -- \\ 2 & + \\ 1 & + \end{array} \qquad \sigma_1\sigma_2\sigma_3\sigma_1$$

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

4		
3		
2	$-$	$\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$
1	$+$	

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Example

$$\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$$

$\overline{\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1}$ is the desired commutation class.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Observation

For any $\overline{\rho} \in \mathcal{C}(w_0)$, the ith row $P_{\overline{\rho}}(i)$ has $(n-i)$ $+$'s and $(i-1)$ $-$'s.

Upper Bound on $|\mathcal{C}(w_0)|$

Lemma

The map $\overline{\rho} \mapsto P_{\overline{\rho}}$ is an injection.

Observation

For any $\overline{\rho} \in \mathcal{C}(w_0)$, the ith row $P_{\overline{\rho}}(i)$ has $(n-i)$ $+$'s and $(i-1)$ $-$'s.

Theorem

The number of commutation classes of w_0 is at most $\prod_{i=1}^{n} \binom{n-1}{n-i}$.

Upper Bound on $|\mathcal{C}(w_0)|$

Theorem

The number of commutation classes of w_0 is at most $\prod_{i=1}^{n} \binom{n-1}{n-i}$.

Taking logarithms,

$$\log \prod_{i=1}^{n} \binom{n-1}{n-i} \leq n \log (n-1)! - 2 \sum_{i=0}^{n-1} \log i!$$

Upper Bound on $|\mathcal{C}(w_0)|$

Theorem

The number of commutation classes of w_0 is at most $\prod_{i=1}^{n} \binom{n-1}{n-i}$.

Taking logarithms,

$$\begin{aligned}
\log \prod_{i=1}^{n} \binom{n-1}{n-i} &\leq n \log (n-1)! - 2 \sum_{i=0}^{n-1} \log i! \\
&\approx n^2 \log (n-1) - 2 \sum_{i=0}^{n-1} i \log i \quad \text{(by Stirling's)}
\end{aligned}$$

Upper Bound on $|\mathcal{C}(w_0)|$

Theorem

The number of commutation classes of w_0 is at most $\prod_{i=1}^{n} \binom{n-1}{n-i}$.

Taking logarithms,

$$\begin{aligned}
\log \prod_{i=1}^{n} \binom{n-1}{n-i} &\leq n \log (n-1)! - 2 \sum_{i=0}^{n-1} \log i! \\
&\approx n^2 \log (n-1) - 2 \sum_{i=0}^{n-1} i \log i \quad \text{(by Stirling's)} \\
&\approx n^2 \log (n-1) - 2 \int_0^{n-1} x \log x \, dx
\end{aligned}$$

Upper Bound on $|\mathcal{C}(w_0)|$

Theorem

The number of commutation classes of w_0 is at most $\prod_{i=1}^{n} \binom{n-1}{n-i}$.

Taking logarithms,

$$\begin{aligned}
\log \prod_{i=1}^{n} \binom{n-1}{n-i} &\leq n \log{(n-1)!} - 2 \sum_{i=0}^{n-1} \log i! \\
&\approx n^2 \log{(n-1)} - 2 \sum_{i=0}^{n-1} i \log i \quad \text{(by Stirling's)} \\
&\approx n^2 \log{(n-1)} - 2 \int_0^{n-1} x \log x \, dx \\
&\approx \frac{1}{2} n^2.
\end{aligned}$$

Upper Bound on $|\mathcal{C}(w_0)|$

Theorem

The number of commutation classes of w_0 is at most $\prod_{i=1}^{n} \binom{n-1}{n-i}$.

Taking logarithms,

$$\begin{aligned}
\log \prod_{i=1}^{n} \binom{n-1}{n-i} &\leq n \log{(n-1)!} - 2\sum_{i=0}^{n-1} \log i! \\
&\approx n^2 \log{(n-1)} - 2\sum_{i=0}^{n-1} i \log i \quad \text{(by Stirling's)} \\
&\approx n^2 \log{(n-1)} - 2\int_0^{n-1} x \log x \, dx \\
&\approx \frac{1}{2} n^2.
\end{aligned}$$

So $|\mathcal{C}(w_0)| \leq e^{n^2/2} = 2^{n^2 \log_2(e)/2}$ and $\frac{1}{2}\log_2(e) \approx 0.7213$.

Upper Bound on $|\mathcal{C}(w_0)|$

What is the constant c in the exponent?

- (Knuth, 1992) $\sim 0.1667 \leq c \leq \sim 0.7924$
- **(Felsner, 1997)** $c \leq \sim \mathbf{0.6974}$
- (Felsner and Valtr, 2011) $\sim 0.1887 \leq c \leq \sim 0.6571$
- (Dumitrescu and Mandal, 2020) $\sim 0.2083 \leq c$

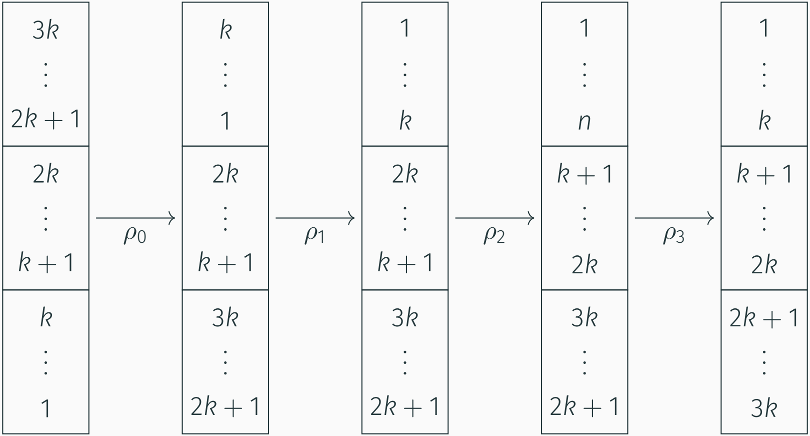
Lower Bound on $|\mathcal{C}(w_0)|$

Idea

Suppose $n = 3k$. Count commutation classes of the form $\overline{\rho_0\rho_1\rho_2\rho_3}$, where

- ρ_0 yields $[\underline{2k+1, \ldots, 3k}, \underline{k+1, \ldots, 2k}, \underline{1, \ldots, k}]$,
- ρ_1 yields $[1, \ldots, k, k+1, \ldots, 2k, \underline{3k, \ldots, 2k+1}]$,
- ρ_2 yields $[1, \ldots, k, \underline{2k, \ldots, k+1}, 2k+1, \ldots, 3k]$,
- ρ_3 yields $[\underline{k, \ldots, 1}, k+1, \ldots, 2k, 2k+1, \ldots, 3k]$.

Lower Bound on $|\mathcal{C}(w_0)|$



Lower Bound on $|\mathcal{C}(w_0)|$

Some notation:

- $B_k := |\mathcal{C}(w_0)|$ where $w_0 \in S_k$
- $w_{3k} := [2k+1, \ldots 3n, k+1, \ldots, 2k, 1, \ldots k] \in S_{3k}$

Lower Bound on $|\mathcal{C}(w_0)|$

Some notation:

- $B_k := |\mathcal{C}(w_0)|$ where $w_0 \in S_k$
- $w_{3k} := [2k+1, \ldots 3n, k+1, \ldots, 2k, 1, \ldots k] \in S_{3k}$

Observation

A lower bound for B_{3k} is $B_{3k} \geq |\mathcal{C}(w_{3k})| \cdot B_k^3$.

Lower Bound on $|\mathcal{C}(w_0)|$

Some notation:

- $B_k := |\mathcal{C}(w_0)|$ where $w_0 \in S_k$
- $w_{3k} := [2k+1, \ldots 3n, k+1, \ldots, 2k, 1, \ldots k] \in S_{3k}$

Observation

A lower bound for B_{3k} is $B_{3k} \geq |\mathcal{C}(w_{3k})| \cdot B_k^3$.

How do we estimate $|\mathcal{C}(w_{3k})|$?

Lower Bound on $|\mathcal{C}(w_0)|$

Some notation:

- $B_k := |\mathcal{C}(w_0)|$ where $w_0 \in S_k$
- $w_{3k} := [2k+1, \ldots 3n, k+1, \ldots, 2k, 1, \ldots k] \in S_{3k}$

Observation

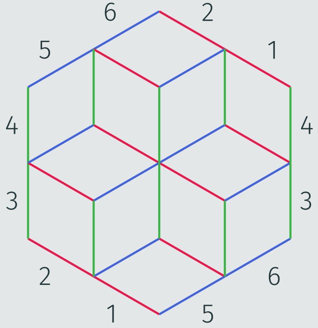
A lower bound for B_{3k} is $B_{3k} \geq |\mathcal{C}(w_{3k})| \cdot B_k^3$.

How do we estimate $|\mathcal{C}(w_{3k})|$?

Count rhombic tilings via Elnitsky's bijection!

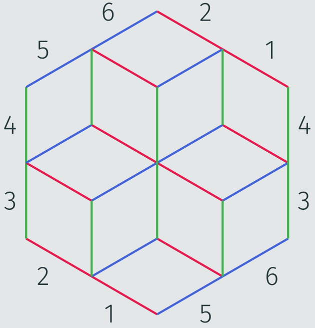
Lower Bound on $|\mathcal{C}(w_0)|$

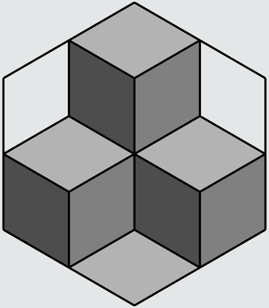
Example (k = 2)



Lower Bound on $|\mathcal{C}(w_0)|$

Example (n = 2)





Lower Bound on $|\mathcal{C}(w_0)|$

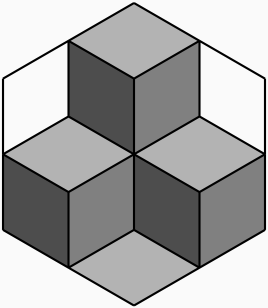
Definition

A plane partition is a matrix of nonnegative integers $\pi_{i,j}$ such that $\pi_{i,j} \geq \pi_{i,j+1}$ and $\pi_{i,j} \geq \pi_{i+1,j}$.

Lower Bound on $|\mathcal{C}(w_0)|$

Definition

A plane partition is a matrix of nonnegative integers $\pi_{i,j}$ such that $\pi_{i,j} \geq \pi_{i,j+1}$ and $\pi_{i,j} \geq \pi_{i+1,j}$.



The plane partition $\begin{matrix} 2 & 1 \\ 1 & 0 \end{matrix}$

Lower Bound on $|\mathcal{C}(w_0)|$

Lemma

The commutation classes of w_{3k} are in bijection with $k \times k$ plane partitions with entries at most k.

Lower Bound on $|\mathcal{C}(w_0)|$

Lemma

The commutation classes of w_{3k} are in bijection with $k \times k$ plane partitions with entries at most k.

Theorem (MacMahon, 1916)

The number of $i \times j$ plane partitions with entries at most k is equal to

$$\prod_{a=0}^{i-1} \prod_{b=0}^{j-1} \prod_{c=0}^{k-1} \frac{a+b+c+2}{a+b+c+1}.$$

Lower Bound on $|\mathcal{C}(w_0)|$

To approximate the number of such $k \times k$ plane partitions with entries at most k, we take logarithms.

$$\log \prod_{0 \le a,b,c \le k-1} \frac{a+b+c+2}{a+b+c+1}$$

Lower Bound on $|\mathcal{C}(w_0)|$

To approximate the number of such $n \times n$ plane partitions with entries at most k, we take logarithms.

$$\begin{aligned} &\log \prod_{0 \le a,b,c \le k-1} \frac{a+b+c+2}{a+b+c+1} \\ = &\sum_{0 \le a,b,c \le k-1} \log(a+b+c+2) - \log(a+b+c+1) \end{aligned}$$

Lower Bound on $|\mathcal{C}(w_0)|$

To approximate the number of such $k \times k$ plane partitions with entries at most k, we take logarithms.

$$
\begin{aligned}
&\log \prod_{0 \le a,b,c \le k-1} \frac{a+b+c+2}{a+b+c+1} \\
&= \sum_{0 \le a,b,c \le k-1} \log(a+b+c+2) - \log(a+b+c+1) \\
&= \sum_{0 \le a,b \le k-1} \log(a+b+n+1) - \log(a+b+1)
\end{aligned}
$$

Lower Bound on $|\mathcal{C}(w_0)|$

To approximate the number of such $k \times k$ plane partitions with entries at most k, we take logarithms.

$$
\begin{aligned}
&\log \prod_{0 \le a,b,c \le k-1} \frac{a+b+c+2}{a+b+c+1} \\
&= \sum_{0 \le a,b,c \le k-1} \log(a+b+c+2) - \log(a+b+c+1) \\
&= \sum_{0 \le a,b \le k-1} \log(a+b+k+1) - \log(a+b+1) \\
&\approx \int_{x=0}^{k-1} \int_{y=0}^{k-1} \log(x+y+k+1) - \log(x+y+1)\, dy\, dx
\end{aligned}
$$

Lower Bound on $|\mathcal{C}(w_0)|$

To approximate the number of such $k \times k$ plane partitions with entries at most k, we take logarithms.

$$
\begin{aligned}
&\log \prod_{0 \le a,b,c \le k-1} \frac{a+b+c+2}{a+b+c+1} \\
&= \sum_{0 \le a,b,c \le k-1} \log(a+b+c+2) - \log(a+b+c+1) \\
&= \sum_{0 \le a,b \le k-1} \log(a+b+k+1) - \log(a+b+1) \\
&\approx \int_{x=0}^{k-1} \int_{y=0}^{k-1} \log(x+y+k+1) - \log(x+y+1)\, dy\, dx \\
&\approx \left(\frac{9}{2} \ln(3) - 6 \ln(2) \right) k^2
\end{aligned}
$$

Lower Bound on $|\mathcal{C}(w_0)|$

To approximate the number of such $k \times k$ plane partitions with entries at most k, we take logarithms.

$$
\begin{aligned}
&\log \prod_{0 \le a,b,c \le k-1} \frac{a+b+c+2}{a+b+c+1} \\
&= \sum_{0 \le a,b,c \le k-1} \log(a+b+c+2) - \log(a+b+c+1) \\
&= \sum_{0 \le a,b \le k-1} \log(a+b+k+1) - \log(a+b+1) \\
&\approx \int_{x=0}^{k-1} \int_{y=0}^{k-1} \log(x+y+k+1) - \log(x+y+1)\, dy\, dx \\
&\approx \left(\frac{9}{2}\ln(3) - 6\ln(2)\right) k^2 \\
&\approx 0.7848 k^2.
\end{aligned}
$$

Lower Bound on $|\mathcal{C}(w_0)|$

To approximate the number of such $k \times k$ plane partitions with entries at most k, we take logarithms.

$$
\begin{aligned}
&\log \prod_{0 \le a,b,c \le k-1} \frac{a+b+c+2}{a+b+c+1} \\
&= \sum_{0 \le a,b,c \le k-1} \log(a+b+c+2) - \log(a+b+c+1) \\
&= \sum_{0 \le a,b \le k-1} \log(a+b+k+1) - \log(a+b+1) \\
&\approx \int_{x=0}^{k-1} \int_{y=0}^{k-1} \log(x+y+k+1) - \log(x+y+1)\, dy\, dx \\
&\approx \left(\frac{9}{2}\ln(3) - 6\ln(2)\right) k^2 \\
&\approx 0.7848k^2.
\end{aligned}
$$

This yields $|\mathcal{C}(w_{3k})| \approx e^{0.7848k^2}$.

Lower Bound on $|\mathcal{C}(w_0)|$

Finally, we derive our lower bound

$$B_{3k} \geq e^{0.7848k^2} \cdot B_k^3$$

Lower Bound on $|\mathcal{C}(w_0)|$

Finally, we derive our lower bound

$$\begin{aligned} B_{3k} &\geq e^{0.7848k^2} \cdot B_k^3 \\ &\geq e^{0.7848k^2} \cdot (e^{0.7848(k/3)^2})^3 \cdot B_{k/3}^{3^2} \end{aligned}$$

Lower Bound on $|\mathcal{C}(w_0)|$

Finally, we derive our lower bound

$$\begin{aligned} B_{3k} &\geq e^{0.7848k^2} \cdot B_k^3 \\ &\geq e^{0.7848k^2} \cdot (e^{0.7848(k/3)^2})^3 \cdot B_{k/3}^{3^2} \\ &\approx \prod_{i=0}^{\lfloor \log_3(k) \rfloor} e^{0.7848k^2/3^i} \end{aligned}$$

Lower Bound on $|\mathcal{C}(w_0)|$

Finally, we derive our lower bound

$$\begin{aligned} B_{3k} &\geq e^{0.7848k^2} \cdot B_k^3 \\ &\geq e^{0.7848k^2} \cdot (e^{0.7848(k/3)^2})^3 \cdot B_{k/3}^{3^2} \\ &\approx \prod_{i=0}^{\lfloor \log_3(k) \rfloor} e^{0.7848k^2/3^i} \\ &\approx e^{0.7848k^2 \cdot 3/2} \end{aligned}$$

Lower Bound on $|\mathcal{C}(w_0)|$

Finally, we derive our lower bound

$$\begin{aligned} B_{3k} &\geq e^{0.7848k^2} \cdot B_k^3 \\ &\geq e^{0.7848k^2} \cdot (e^{0.7848(k/3)^2})^3 \cdot B_{k/3}^{3^2} \\ &\approx \prod_{i=0}^{\lfloor \log_3(k) \rfloor} e^{0.7848k^2/3^i} \\ &\approx e^{0.7848k^2 \cdot 3/2} \end{aligned}$$

Thus, $\log_2(B_{3k}) \approx \log_2(e) \cdot 0.7848 \cdot \frac{3}{2} \cdot \frac{1}{3^2} k^2 \approx 0.1887k^2$.

Lower Bound on $|\mathcal{C}(w_0)|$

What is the constant c in the exponent?

- (Knuth, 1992) $\sim 0.1667 \le c \le \sim 0.7924$
- (Felsner, 1997) $c \le \sim 0.6974$
- **(Felsner and Valtr, 2011)** $\boldsymbol{\sim 0.1887 \le c} \le\sim 0.6571$
- (Dumitrescu and Mandal, 2020) $\sim 0.2083 \le c$

Open Problems

- Can a better lower bound be obtained via a decomposition into 4 blocks?

Open Problems

- Can a better lower bound be obtained via a decomposition into 4 blocks?
- Is there a criterion for realizable weaving patterns?

Open Problems

- Can a better lower bound be obtained via a decomposition into 4 blocks?
- Is there a criterion for realizable weaving patterns?
- How are the sizes of commutation classes distributed?

Thank You!