Problem Set 4 (due Monday, May 7th)

**A:** Let $\theta = \sqrt{2+\sqrt{2}}$. Let $K = \mathbb{Q}(\theta)$. Show that $K/\mathbb{Q}$ is a Galois extension.

**B:** Let $M$ be the splitting field for $f(x) = (x^2-2)(x^2-3)(x^2-5)$ over $\mathbb{Q}$. Let $K$ be the splitting field for $g(x) = x^{120} - 1$ over $\mathbb{Q}$. Prove that $M \subseteq K$. Furthermore, prove that $M \neq K$.

**C:** Let $n \geq 1$. Let $\zeta_n = e^{\frac{2\pi i}{n}}$. Suppose that $F$ is a subfield of $\mathbb{C}$ and that $\mathbb{Q}(\zeta_n) \subseteq F$. Suppose that $\gamma \in F$. Let $\beta$ denote one of the roots of the polynomial $f(x) = x^n - \gamma$ in $\mathbb{C}$. Let $K$ denote the splitting field for $f(x)$ over $F$. Prove that $K = F(\beta)$. Let $G = Gal(K/F)$. Prove that $G$ is a cyclic group and that $|G|$ divides $n$.

**D.** The Eisenstein criterion implies that the polynomial $f(x) = x^5 - 2$ is irreducible over $\mathbb{Q}$. Let $K$ denote the splitting field for $f(x)$ over $\mathbb{Q}$. Let $F$ denote the splitting field for $x^5 - 1$ over $\mathbb{Q}$.

(a) Prove that $F \subseteq K$.

(b) Prove that $[K : \mathbb{Q}] = 20$.

(c) Prove that $f(x)$ is irreducible over $F$ and that $K$ is the splitting field for $f(x)$ over $F$.

(d) Prove that $Gal(K/F)$ is a cyclic group of order 5.