Some Practice Questions for the Math 404 Midterm

1. Consider $K = \mathbb{Q}(\sqrt{2},\ \sqrt{-2},\ \sqrt[4]{5})$. Prove that there exists a polynomial $g(x) \in \mathbb{Q}[x]$ with the following property:

$$K \text{ is the splitting field for } g(x) \text{ over } \mathbb{Q}.$$

Justify your answer carefully.

2. Give an example of two subfields F and L of $\mathbb{C}$. with the following properties:

$$[F : \mathbf{Q}] = [L : \mathbf{Q}] = 5, \quad F \text{ and } L \text{ are isomorphic as fields, but } F \neq L.$$

Justify your answer carefully.

3. Consider the polynomial $f(x) = x^3 + x + 3$ in $\mathbb{Q}[x]$. This polynomial has three distinct roots in $\mathbb{C}$, but only one root in $\mathbb{R}$. (You may use this fact in this question. Don't bother to verify it.) Suppose that β_1 is the root in $\mathbb{R}$ and that β_2 and β_3 are the roots which are not in $\mathbb{R}$.

(a) Prove that the fields $\mathbb{Q}(\beta_1)$, $\mathbb{Q}(\beta_2)$, and $\mathbb{Q}(\beta_3)$ are isomorphic to each other.

(b) Prove that $\mathbb{Q}(\beta_1) \neq \mathbb{Q}(\beta_2)$.

(c) Prove that $\mathbb{Q}(\beta_3) \neq \mathbb{Q}(\beta_2)$.

(d) Find the minimal polynomial for β_2 over $\mathbb{Q}$. Justify your answer.

(e) Find the minimal polynomial for β_1 over $\mathbb{R}$. Justify your answer.

(f) Find the minimal polynomial for β_2 over $\mathbb{R}$. Justify your answer.

4. This question concerns the following complex numbers:

$$\zeta_5 = cos(2\pi/5) + sin(2\pi/5)i \qquad and \qquad \beta = (\zeta_5 - 1)^{-4} \quad .$$

Does there exists a polynomial $f(x) \in \mathbb{Q}[x]$ with the following property: $f(\zeta_5) = \beta$?

Justify your answer carefully.

5. Let $\theta = \sqrt{2}+i$. Consider the field $K = \mathbb{Q}(\theta)$. Carefully determine $[K : \mathbb{Q}]$. Furthermore, prove that every element of $Aut(K/\mathbb{Q})$ has order 1 or 2.