Problem Set 6 (due Friday, May 25th)

A: Find all subfields of $\mathbb{Q}(\zeta_8)$.

B: Let K be the splitting field over $\mathbb{Q}$ for $f(x) = x^4 - 2$. We know that $K = \mathbb{Q}(\sqrt[4]{2}, i)$ and that $[K : \mathbb{Q}] = 8$. Determine all subfields F of K such that $[F : \mathbb{Q}] = 4$. In addition, determine which of those subfields F is a Galois extension of $\mathbb{Q}$.

C: Suppose that $r \in \mathbb{Q}$. Let $\beta = cos(r\pi)$. Prove that β is algebraic over $\mathbb{Q}$. Let $K = \mathbb{Q}(\beta)$. Prove that $\mathbb{Q}(\beta)$ is a Galois extension of $\mathbb{Q}$ and that $Gal(K/\mathbb{Q})$ is an abelian group.

D: Let $K = \mathbb{Q}(\omega)$, where $\omega = cos(\frac{2\pi}{17}) + sin(\frac{2\pi}{17})i$. Prove that K contains a unique subfield L such that $[L : \mathbb{Q}] = 8$. Prove that L is a Galois extension of $\mathbb{Q}$. Find an element $\beta \in L$ such that $L = \mathbb{Q}(\beta)$.

E: Prove the existence of a Galois extension K of $\mathbb{Q}$ such that $Gal(K/\mathbb{Q}) \cong \mathbb{Z}/14\mathbb{Z}$.

F: Prove that $\sqrt{2} \notin \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Prove that K is a Galois extension of $\mathbb{Q}$, that $[K : \mathbb{Q}] = 8$, and that

$$Gal(K/\mathbb{Q}) \ \cong \ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ \ .$$

G. Suppose that K is a finite Galois extension of $\mathbb{Q}$ and that $Gal(K/\mathbb{Q}) \cong S_3$. Prove that there exists a polynomial $g(x) \in \mathbb{Q}[x]$ such that $g(x)$ has degree 3 and K is the splitting field for $g(x)$ over $\mathbb{Q}$.

H: Suppose that E and F are finite extensions of $\mathbb{Q}$ and that $E \cap F = \mathbb{Q}$. By the Primitive Element Theorem, we know that $E = \mathbb{Q}(\alpha)$ and $F = \mathbb{Q}(\beta)$ for certain elements $\alpha \in E$ and $\beta \in F$. Let $K = \mathbb{Q}(\alpha,\ \beta)$.

TRUE OR FALSE: $[K : \mathbb{Q}] = [E : \mathbb{Q}][F : \mathbb{Q}]$.

If true, give a proof. If false, give a counterexample.