Problem Set 5 (due Monday, May 14th)

A: This question concerns the following polynomial:

$$g(x) \;=\; x^7 - 12x^6 + 39x^5 + 21x^4 - 27x^3 + 3x + 6 \quad .$$

Let θ denote one of the roots of $g(x)$ in $\mathbb{C}$. Let $K = \mathbb{Q}(\theta)$. Consider the following two elements of the field K:

$$\gamma \;=\; 2 + 3\theta + 6\theta^2 - 7\theta^4 \;, \qquad\qquad \delta \;=\; \frac{4 + 7\theta + 2\theta^2 - 4\theta^5}{3 + 8\theta + 3\theta^3 - 5\theta^4 + 2\theta^6} \quad .$$

Prove carefully that there exists a polynomial $f(x)$ in $\mathbb{Q}[x]$ such that $f(\gamma) = \delta$.

B: Suppose that K is a finite Galois extension of $\mathbb{Q}$ and that $Gal(K/\mathbb{Q}) \cong S_5$. Show that K cannot contain $\sqrt[5]{2}$.

C: Suppose that $f(x) \in \mathbb{Q}[x]$ and that $f(x)$ has degree 4. Let K be the splitting field for $f(x)$ over $\mathbb{Q}$. Suppose that $Gal(K/\mathbb{Q}) \cong S_4$. Prove that $f(x)$ is irreducible over $\mathbb{Q}$.

D: Prove that there exist four complex numbers $\beta_1, \beta_2, \beta_3, \beta_4$ satisfying the following four equations:

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 \;=\; 1, \qquad \beta_1\beta_2\beta_3 + \beta_1\beta_2\beta_4 + \beta_1\beta_3\beta_4 + \beta_2\beta_3\beta_4 \;=\; 1$$

$$\beta_1\beta_2 + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_3 + \beta_2\beta_4 + \beta_3\beta_4 \;=\; 1, \qquad \beta_1\beta_2\beta_3\beta_4 = 1 \quad .$$

Furthermore, show that $\beta_1, \beta_2, \beta_3, \beta_4$ are algebraic over $\mathbb{Q}$. Let $K = \mathbb{Q}(\beta_1, \beta_2, \beta_3, \beta_4)$. Determine $[K : \mathbb{Q}]$. Show that $K = \mathbb{Q}(\beta_1)$.

E:. This is a continuation of problem **D** on problem set 4. That question concerned the polynomial $f(x) = x^5 - 2$. Recall that K denotes the splitting field for $f(x)$ over $\mathbb{Q}$ and that F denotes the splitting field for $x^5 - 1$ over $\mathbb{Q}$. Recall also that $\mathbb{Q} \subset F \subset K$ and that $[K : \mathbb{Q}] = 20$. We know that K is a finite Galois extension of $\mathbb{Q}$. Let $G = Gal(K/\mathbb{Q})$ and let $N = Gal(K/F)$. Prove that N is a normal subgroup of G, that N is a cyclic group of order 5, and that G/N is a cyclic group of order 4. Furthermore, prove that G is a nonabelian group.