Problem Set 3 (due Monday, April 23rd)

A: Let $K = \mathbb{Q}(\zeta_5)$. We proved in class that $[K : \mathbb{Q}] = 4$ and that the minimal polynomial for ζ_5 over $\mathbb{Q}$ is $m(x) = x^4 + x^3 + x^2 + x + 1$. It follows that the set $\{ 1, \zeta_5, \zeta_5^2, \zeta_5^3 \}$ is a basis for K over $\mathbb{Q}$. We also proved that $\sqrt{5} \in K$. Therefore, there exists $a, b, c, d \in \mathbb{Q}$ such that

$$\sqrt{5} = a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3$$

and that a, b, c, d are unique. Find a, b, c, d.

B: Prove that there exist infinitely many distinct extensions K of $\mathbb{Q}$ such that $[K : \mathbb{Q}] = 2$.

C: Suppose that α is a root in $\mathbb{C}$ of the polynomial $f(x) = x^8 + 4x^7 + 2x^3 - 6x + 2$. Suppose that β is a root in $\mathbb{C}$ of the polynomial $g(x) = x^9 + 3x^6 + 12x^2 - 6x + 15$. Let $F = \mathbb{Q}(\alpha)$. Let $K = \mathbb{Q}(\beta)$. Prove that $F \cap K = \mathbb{Q}$.

D: Let K be the splitting field over $\mathbb{Q}$ for $x^6 - 8$. Prove that $K = \mathbb{Q}(\sqrt{2}, \sqrt{-3})$ and that $[K : \mathbb{Q}] = 4$.

E: Let $F = \mathbb{Q}(\sqrt[4]{2})$. Determine the cardinality of the set $Emb_{\mathbb{Q}}(F, \mathbb{C})$. Determine the order of the group $Aut(F/\mathbb{Q})$. How many distinct subfields of $\mathbb{C}$ are isomorphic to F?