Problem Set 2 (due Monday, April 16th)

A: This problem concerns the complex number $\omega = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

(a) Show that $\mathbb{Q}(\omega) = \mathbb{Q})(\sqrt{2},\ i)$.

(b) Determine $[\mathbb{Q}(\omega)\ :\ \mathbb{Q}]$.

(c) Verify that $\omega^8 = 1$, but $\omega^4 \neq 1$. Find the minimal polynomial for ω over $\mathbb{Q}$.

(d) Find the minimal polynomial for ω over $\mathbb{Q}(i)$.

B. Let K be the splitting field for $x^4 - 2$ over $\mathbb{Q}$. Determine $[K : \mathbb{Q}]$. Furthermore, show that K contains $\mathbb{Q}(\omega)$, where ω is the number specified in problem **A**. Is $K = \mathbb{Q}(\omega)$?

C. Suppose that K is a subfield of $\mathbb{C}$ and that $[K : \mathbb{Q}] = 2$. Prove that there exists an integer d with the following property: K is the splitting field for $x^2 - d$ over $\mathbb{Q}$.

D. Consider the polynomial $f(x) = (x^2 + x + 1)(x^2 + x - 1)(x - 2)$. Thus, $f(x) \in \mathbf{Q}[x]$. The polynomial $f(x)$ has five distinct roots in $\mathbb{C}$. Let F be the splitting field for $f(x)$ over $\mathbf{Q}$. Prove that every element of the group $Aut(F/\mathbf{Q})$ has order 1 or 2.

E. Suppose that p is a prime. Let $F = \mathbb{Z}/p\mathbb{Z}$. Suppose K is a finite extension of F. Let $n = [K : F]$. We know that K is a finite field with p^n elements. Define a map $\sigma : K \longrightarrow K$ by $\sigma(k) = k^p$ for all $k \in K$. As explained in class, σ is an automorphism of the field K. Prove that σ has order n in the group $Aut(K)$.

F. Let F be a field and let K be an extension of F. Let $G = Aut(K/F)$. Let H be a subgroup of G. Let

$$L = \{\ \alpha \bigm| \alpha \in K \ \ and \ \ \varphi(\alpha) = \alpha \ \ for\ all\ \varphi \in H\ \ \}\ .$$

Prove that L is a subfield of K and that $F \subseteq L$.

G. Suppose F is a field and that K is a finite extension of F. Suppose that $[K : F] = p$, where p is a prime number. Suppose that $\beta \in K$, but $\beta \notin F$. Prove that $K = F(\beta)$.