Practice Questions for the Final Exam

**Question 1.** Find *all* solutions to the congruence $13x \equiv 12 \pmod{35}$. Also, answer the following questions about the solutions to the above congruence. Are there solutions $x$ such that $x$ gives a remainder of 1 when divided by 48? Are there solutions $x$ such that $x$ gives a remainder of 1 when divided by 49?

**Question 2.** This question concerns the Diophantine equation $x^2 - 35y^2 = 11$. The purpose of this question is to prove that this Diophantine equation has no solutions. Suppose to the contrary that $x = a$ and $y = b$ is a solution to that equation, where $a,\ b \in \mathbf{Z}$.
(a) Prove that $gcd(a, 11) = 1$ and $gcd(b, 11) = 1$.

(b) Show that $35b^2 \equiv a^2 \pmod{11}$. By raising both sides of this congruence to a certain power, show that you can obtain a contradiction.

**Question 3.** Find all solutions to the congruence $x^5 \equiv 1 \pmod{11213}$. (Remark: You may use the fact that 11213 is a prime.)

**Question 4.** Suppose that $a, b, c \in \mathbf{Z}$ and that $m$ is a positive integer. Assume that $gcd(a, m) = 1$ and $gcd(b, m) = 1$. Assume that $c \equiv ab \pmod{m}$. Prove that $gcd(c^3, m) = 1$.

**Question 6.** Find *all* primes $p$ such that $ord_p(-5) = 2$. Find *all* primes $p$ such that $ord_p(-5) = 3$. Find *all* primes $p$ such that $ord_p(3) = 8$.

**Question 7.** This question concerns the integer $n = 100000001$. Prove that if $p$ is a prime which divides $n$, then $p \equiv 1 \pmod{16}$. The smallest prime $p$ satisfying $p \equiv 1 \pmod{16}$ is $p = 17$. Prove that $17 \big| n$.

**Question 8.** Find the remainder when 185648291 is divided by 99.

**Question 9.** Find the remainder when $2^{19}5^8 13^{11} 29^{18} - 3^{600}$ is divided by 7.