NUMBER THEORY PROBLEM SET 2 (due Friday, April 13th)

A: There exist primes p such that $p + 6k$ is also prime for $k = 1, 2$ and 3. One such prime is $p = 11$. Another such prime is $p = 41$. Prove that there exists exactly one prime p such that $p + 6k$ is also prime for $k = 1, 2, 3$ and 4.

B: Prove that if $n \in \mathbf{Z}$, then n^3 gives a remainder of 0, 1, or 6 when divided by 7.

C: Prove that if n is an odd integer, then n is of the form $4k + 1$ or $4k + 3$, where $k \in \mathbf{Z}$.

D: Prove that if n is an odd integer, then n^2 gives a remainder of 1 when divided by 8.

E: Prove that if n is an even integer, then n^2 gives a remainder of 0 or 4 when divided by 8.

F: Let $d = gcd(1189,\ 215)$. Determine d. Furthermore, find integers m and n such that

$$1189m + 215n = d \quad .$$

G: Consider the equation $x^2 - 6y^2 = 10$. It turns out that this equation has infinitely many solutions where $x, y \in \mathbf{Z}$. Suppose that $x = a,\ y = b$ is one of those solutions. Prove that $(a, b) = 1$.