Important propositions about ring homomorphisms and ideals.

In the following propositions, the letters R and S will always denote rings.

1. Suppose that $\varphi : R \to S$ is a ring homomorphism. Then $Ker(\varphi)$ is an ideal in the ring R and $Im(\varphi)$ is a subring of the ring S. (Note: We often denote $Im(\varphi)$ by $\varphi(R)$.)

2. (The First Isomorphism Theorem.) Suppose that $\varphi : R \to S$ is a ring homomorphism. Let $I = Ker(\varphi)$. Define a map ψ from R/I to $\varphi(R)$ by

$$\psi(r + I) \;=\; \varphi(r)$$

for all $r \in R$. Then ψ is an isomorphism from the quotient ring R/I to the ring $\varphi(R)$. Hence, the rings R/I and $\varphi(R)$ are isomorphic.

3. (The Correspondence Theorem.) Suppose that $\varphi : R \to S$ is a surjective ring homomorphism. Let $K = Ker(\varphi)$. There is a 1-1 correspondence between the following sets:

$$\{\ I \bigm| I \textit{ is an ideal of } R \textit{ containing } K\ \} \qquad \textit{and} \qquad \{\ J \bigm| J \textit{ is an ideal of } S\ \}$$

defined by letting I correspond to $\varphi(I) = \{\ \varphi(i) \bigm| i \in I\ \}$ and letting J correspond to $\varphi^{-1}(J) = \{\ r \in R \mid \varphi(r) \in J\}$. Furthermore, if I_1 and I_2 are ideals of R containing K and if J_1 and J_2 are the corresponding ideals of S, then $I_1 \subseteq I_2$ if and only if $J_1 \subseteq J_2$.

4. Suppose that $\varphi : R \to S$ is a surjective ring homomorphism. Let $K = Ker(\varphi)$. Suppose that I is an ideal of R containing K. Let $J = \varphi(I)$. Then

$$R/I \;\cong\; S/J.$$

One obtains an isomorphism χ from R/I to S/J by defining

$$\chi(r + I) \;=\; \varphi(r) + J$$

for all $r \in R$.

5. Suppose that R and S are rings with unity and that $\varphi : R \to S$ is a surjective ring homomorphism. Then $\varphi(1_R) = 1_S$.

6. Suppose that R and S are integral domains and that $\varphi : R \to S$ is a ring homomorphism. Then either $\varphi(1_R) = 0_S$ or $\varphi(1_R) = 1_S$. If $\varphi(1_R) = 0_S$, then $\varphi(r) = 0_S$ for all $r \in R$.

7. Suppose that R is a commutative ring with unity. Let I be an ideal of R. Then I is a prime ideal of R if and only if R/I is an integral domain.

8. Suppose that R is a commutative ring with unity. Let I be an ideal of R. Then I is a maximal ideal of R if and only if R/I is a field.

9. Suppose that R is a commutative ring with unity. If I is a maximal ideal of R, then I is a prime ideal of R.

10. Suppose that R is a ring with unity. Suppose that I is an ideal of R and that $I \neq R$. Then there exists at least one maximal ideal of R such that $I \subseteq M$.

11. Suppose that I and J are ideals in R. Then $I + J$ is an ideal of R. The ideal $I + J$ contains both I and J. Furthermore, if K is any ideal of R containing both I and J, then K contains $I + J$.

12. Suppose that I and J are ideals in R. Then $I \cap J$ is an ideal of R. The ideal $I \cap J$ is contained in both I and J. Furthermore, if K is any ideal of R which is contained in both I and J, then K is contained in $I \cap J$.

13. (The Chinese Remainder Theorem.) Suppose that R is a ring with unity. Suppose that I and J are ideals of R and that $I + J = R$. Let $K = I \cap J$. Define the map

$$\varphi : \; R/K \;\longrightarrow\; (R/I) \oplus (R/J)$$

by $\varphi(r + K) \;=\; (r + I,\; r + J)$ for all $r \in R$. Then φ is a ring isomorphism from R/K to $(R/I) \oplus (R/J)$.

14. Suppose that R is a commutative ring with unity. Suppose that $a \in R$. Let

$$I \;=\; aR \;=\; \{\; ar \;\big|\; r \in R \;\} \;.$$

Then I is an ideal of R containing a. Furthermore, if J is any ideal of R containing a, then $I \subseteq J$. (Note: The ideal aR is referred to as the principal ideal of R generated by a.)

15. Suppose that R is a commutative ring with identity. Suppose that $a,\; b \in R$. Then $aR \subseteq bR$ if and only if there exists an element $r \in R$ such that $a = br$.

16. Suppose that R is an integral domain. Suppose that $a,\; b \in R$. Then $aR = bR$ if and only if there exists an element $u \in U(R)$ such that $a = bu$.