The Fundamental Lemma. *Suppose that F is a field and that m and n are positive integers. Consider a homogeneous system S of m linear equations in n unknowns $x_1, ..., x_n$ with coefficients in F. If $n > m$, then the system S has a solution $x_1 = c_1, ..., x_n = c_n$, where $c_1, ..., c_n \in F$ and $c_j \neq 0_F$ for at least one j, $1 \leq j \leq n$.*

Proof. The system S takes the following form:

$$(\mathrm{S}) \qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0_F \\ \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = 0_F \end{cases}$$

where $a_{ij} \in F$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$. Our proof will be by mathematical induction on m.

The result is easily verified when $m = 1$. In that case, $n > 1$. If $a_{11} = 0_F$, one can take $x_1 = 1_F$, $x_2 = ... = x_n = 0_F$. If $a_{11} \neq 0_F$, one can take $x_2 = ... = x_n = 1_F$ and $x_1 = -a_{11}^{-1}(a_{12} + ... + a_{1n})$. The values specified are clearly in F and at least one of those values is equal to 1_F and hence nonzero.

Now we assume that $m > 1$ and make the inductive hypothesis that the fundamental lemma is true for any homogeneous system of $m - 1$ linear equations over F in more than $m - 1$ unknowns.

We will first consider the case where $a_{i1} = 0_F$ for all i, $1 \leq i \leq m$. That case is quite easy and we don't need the inductive hypothesis. We can choose $x_1 = 1_F$ and $x_j = 0_F$ for $1 < j \leq n$. All the equations are then clearly satisfied.

Now we consider the case where at least one of the coefficients a_{i1} is nonzero. We can rearrange the order of the above equations so that the coefficient for x_1 in the first equation is nonzero. That is, we will now assume that $a_{11} \neq 0_F$. We will use the inductive hypothesis at some point. In the system S, for each i such that $2 \leq i \leq m$, we change the i-th equation by multiplying the first equation by $a_{i1}a_{11}^{-1}$ and subtracting the resulting equation from the i-th equation. This will give a new system of equations which we will call S'. Since these operations can be reversed, the solutions to system S' will be the same as the solutions to system S. System S' takes the following form:

$$
(S') \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0_F \\ 0_F x_1 + b_{22}x_2 + ... + b_{2n}x_n = 0_F \\ \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \\ 0_F x_1 + b_{m2}x_2 + ... + b_{mn}x_n = 0_F \end{cases}
$$

where $b_{ij} \in F$ for all i, j, $2 \leq i \leq m$, $2 \leq j \leq n$. Consider the following homogeneous system of linear equations:

$$
(T) \quad \begin{cases} b_{22}x_2 + ... + b_{2n}x_n = 0_F \\ \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot \qquad \cdot \\ b_{m2}x_2 + ... + b_{mn}x_n = 0_F \end{cases}
$$

This system T is a homogeneous system of $m-1$ equations in the $n-1$ unknowns $x_2, ..., x_n$ with coefficients in F. Note that $n-1 > m-1$ because $n > m$. By the inductive hypothesis, we can assert that system T has a solution of the form $x_2 = c_2, ..., x_n = c_n$, where $c_2, ..., c_n \in F$ and where $c_j \neq 0_F$ for at least one j, $2 \leq j \leq n$. To obtain a solution for the system S', we must just consider the first equation in that system. We want to find $c_1 \in F$ so that

$$
a_{11}c_1 \; + \; \left(a_{12}c_2 + ... + a_{1n}c_n\right) \; = \; 0_F \quad .
$$

We can take

$$
c_1 \; = \; -\, a_{11}^{-1}\left(a_{12}c_2 + ... + a_{1n}c_n\right)
$$

which is clearly an element of F. (Recall that $a_{11} \neq 0_F$.) Therefore, $c_1, ..., c_n \in F$, $c_j \neq 0_F$ for at least one j, $1 \leq j \leq n$, and $x_1 = c_1, ..., x_n = c_n$ is a solution to the system S' and hence to the system S. This completes the proof of the inductive step.

The proof is now complete by the principle of mathematical induction.