FIELD EXTENSIONS AND THEIR DEGREES

Notation and terminology: Suppose that F is a subfield of a field K. We then say that K is an extension of F. If K is finitely generated as a vector space over F, then we say that *"K is a finite extension of F."* We denote $dim_F(K)$ by $[K : F]$. We call $[K : F]$ the *"degree of K over F."* In the following propositions, F, K, and L will always denote fields.

PROPOSITIONS

1: Suppose that K is a finite extension of F and that $\beta \in K$. Then β is algebraic over F.

2: Suppose that F is a subfield of a field K and that $\beta \in K$. Then the following statements are equivalent:

(a) β is algebraic over F.

(b) $F(\beta) = F[\beta]$.

(c) $F(\beta)$ is a finite extension of F.

Furthermore, if β is algebraic over F and $m(x)$ is the minimal polynomial for β over F, then $[F(\beta) : F] = deg(\ m(x)\ )$.

3: Suppose that F is a subfield of a field K, that $\beta_1, ..., \beta_t \in K$, and that $\beta_1, ..., \beta_t$ are all algebraic over F. Then $F(\beta_1, ..., \beta_t) = F[\beta_1, ..., \beta_t]$. Furthermore, $F(\beta_1, ..., \beta_t)$ is a finite extension of F.

4: Suppose that F is a subfield of a field K and that K is a subfield of a field L. Then L is a finite extension of F if and only if L is a finite extension of K and K is a finite extension of F. If L is a finite extension of F, then

$$[L : F] \ = \ [L : K][K : F].$$

5: Suppose that F is a subfield of a field K, that K is a subfield of a field L, and that L is a finite extension of F. Then both $[L : K]$ and $[K : F]$ divide $[L : F]$. Furthermore,

$$K = L \quad \Longleftrightarrow \quad [L : K] = 1 \quad \Longleftrightarrow \quad [K : F] = [L : F] \ .$$

6: Suppose that K is a finite extension of F and that $\beta \in K$. Let $m(x)$ denote the minimal polynomial for β over F. Then $deg(\ m(x)\ )$ divides $[K : F]$. Furthermore, $deg(\ m(x)\ ) = [K : F]$ if and only if $K = F(\beta)$.