The discriminant of a polynomial

Suppose that $f(x)$ is a monic polynomial with coefficients in a field F. Let $n = deg(f(x))$. Assume that $n \geq 2$. Suppose that M is a field containing F and that

$$f(x) = \prod_{i=1}^{n}(x - \theta_i)$$

where $\theta_1, ..., \theta_n \in M$. Let $K = F(\theta_1, ..., \theta_n)$, the splitting field for $f(x)$ in M.

Definition: The discriminant of $f(x)$ is defined by:

$$d = disc(f(x)) = \prod_{1 \leq i < j \leq n} (\theta_i - \theta_j)^2 .$$

Fact: *The discriminant d of $f(x)$ is an element of the field F.*

Note that $d = 0_F$ if and only if $f(x)$ has at least one multiple root in M. We will assume from here on that all the roots of $f(x)$ are distinct and so $d \neq 0_F$.

Let $\gamma = \prod_{1 \leq i < j \leq n} (\theta_i - \theta_j)$. Note that $\gamma \in K$, but $\gamma^2 = d \in F$. Thus $F(\gamma)$ is a subfield of K and $[F(\gamma) : F] = 1$ or 2. The field $F(\gamma)$ is the splitting field in M for the quadratic polynomial $x^2 - d$.

Let $B = \{\theta_1, ..., \theta_n\}$. Consider the group homomorphism $\rho : Aut(K/F) \longrightarrow S_B$ defined by

$$\rho(\sigma) = \sigma|_B .$$

We will identify S_B with the symmetric group S_n. (Recall that we are assuming that $\theta_1, ..., \theta_n$ are all distinct.) Thus we can regard ρ as a group homomorphism from $Aut(K/F)$ to S_n. This homomorphism is injective, but not necessarily surjective.

We assume from here on that F is not of characteristic 2. If $\sigma \in Aut(K/F)$, then $\sigma(\gamma) = \pm\gamma$. More precisely, $\sigma(\gamma) = \gamma$ if $\rho(\sigma)$ is an even permutation, $\sigma(\gamma) = -\gamma$ if $\rho(\sigma)$ is an odd permutation.

If $\gamma \in F$, then d is a square in F and $Im(\rho) \subseteq A_n$. If $\gamma \notin F$, then d is not a square in F and $Im(\rho)$ contains at least one odd permutation.

Formulas for the discriminant in terms of the coefficients of $f(x)$:

If $n = 2$, write $f(x) = x^2 + a_1x + a_2$. Then $disc(f(x)) = a_1^2 - 4a_2$.

If $n = 3$, write $f(x) = x^3 + a_1x^2 + a_2x + a_3$. Then

$$disc(f(x)) = a_1^2a_2^2 - 4a_2^3 - 4a_1^3a_3 - 27a_3^2 + 18a_1a_2a_3 .$$

For $n = 3$, if $f(x) = x^3 + bx + c$, then the discriminant of $f(x)$ is given by the simpler formula

$$disc(f(x)) = -4b^3 - 27c^2 .$$

In general, the following formula is sometimes useful:

$$disc(f(x)) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} f'(\theta_i)$$