Assignment 5.

Given April 8, due April 22.

Objectives: To solve an interesting set of ODE's, to explore questions of stability.

(1) (Description of physical problem taken from notes of M. Overton) This question concerns the motion of a satellite around the sun: the satellite could be a planet, a comet or an unpowered spacecraft, whose mass is assumed to be negligible compared with the mass of the sun. The motion of such a satellite takes place in a plane, so we can describe its coordinates by just two space dimensions x and y. Assume that the sun is at the origin $(0,0)$. The system of differential equations describing the satellite's motion is Newton's law of gravitation, discovered by Newton in the late 17th century, namely

$$x'' = \frac{-x}{(\sqrt{x^2+y^2})^3}, \quad y'' = \frac{-y}{(\sqrt{x^2+y^2})^3}.$$

(a.) This system of two second-order differential equations can be simplified to a system of four first-order differential equations by introducing two new variables $z = x'(t)$ (the speed of the satellite in the x space direction) and $w = y'(t)$ (the speed of the satellite in the y space direction). Write down this system of equations.

(b.) Solve the system you wrote down in (a.) using the fourth order Runge-Kutta method. Start with $x(0) = 4$, $y(0) = 0$, $z(0) = 0$, and $w(0) = .5$. Take $h = .25$ and run out to $tmax = 50$. Plot the position of the satellite at each time step. You should see an almost circular orbit. If you do not, then there is a bug in your code. [Note: If you plot things in MATLAB, be sure to say **axis('square')**, or the axes will be different lengths and your circle will look like an ellipse! Also, if the axes run from $xmin$ to $xmax$ and from $ymin$ to $ymax$, then you will want to have $ymax - ymin = xmax - xmin$, so that the units of distance in x and y are the same. This can be accomplished by typing **axis([xmin xmin+dist ymin ymin+dist])**, where $xmin$ and $ymin$ are the desired minimum values and $dist$ is the number of units that should be represented on each axis.]

Keeping $x(0)$, $y(0)$, and $z(0)$ fixed, try varying $w(0)$. You will have to adjust h for accuracy and $tmax$ so that you see a complete orbit. Some recommended values are as follows:

$w(0)$	h	$tmax$
.5	.25	50
.6	.5	150
.8	.5	200
.4	.25	35
.2	.25	30
.2	.05	30

For initial velocities slightly larger than .5, you should see an elliptical orbit. The fact that planets and comets have elliptical orbits was observed by Kepler in the early 17th century, but it was not explained until Newton developed his law of gravitation 60 years later. The eccentricity of the ellipse depends on the size of the initial velocity $w(0)$. The orbits of planets, such as the Earth, are typically not very eccentric. They look almost like circles. But the orbits of comets, such as Halley's comet and Hale-Bopp, can be very eccentric. That is why Halley's comet can be seen from Earth only once every 76 years and Hale-Bopp about once every 10000 years: the rest of the time they are very far from the sun traveling around their highly eccentric orbits.

If you make the initial velocity large enough, the orbit becomes a hyperbola, with the satellite approaching the direction of a straight line heading right out of the solar system; in this case the satellite escapes the gravitational attraction of the sun. Many comets pass through the solar system on hyperbolic orbits: they enter the solar system, pass by the sun once and then leave the solar system again. Only one spacecraft launched from Earth has ever left the solar system on a hyperbolic orbit; that is Voyager II, which was launched in the late 1970's, flew by Uranus in 1986, and is now on its way out of the solar system, never to return. The borderline case between elliptical and hyperbolic orbits is when $w(0)$ is exactly equal to the *escape velocity*: the value just large enough for the satellite to escape from the solar system. In this case the orbit is actually a parabola, which is the borderline case between an ellipse and a hyperbola.

For initial velocities less than .5 but greater than 0, the orbit is again an ellipse. However, if you make $w(0)$ too small, you may find that your plots show the satellite spiraling into the sun and then swinging off out of the solar system. This is a numerical effect; it is not physical. With a sufficiently small step size, you should be able to see the elliptical orbit for any $w(0) > 0$, but the computation might take too long.

Turn in plots showing the types of orbits described, labelled with the value of $w(0)$ which gives rise to that path.

(c.) Try using MATLAB routine `ODE45` or some other software package for solving initial value problems for ODE's. Describe the method used by the package you find and see if it can obtain accurate solutions for small values of $w(0)$ in a reasonable amount of time. Comment on advantages or disadvantages of this package over your own code.

(2) Consider the system of equations

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1000 & 1 \\ 0 & -1/10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x(0) = 1, \quad y(0) = 2$$

Try using your Runge-Kutta code to solve this system of equations, integrating out to $t = 1$. What size time step is necessary to achieve a reasonably accurate approximate solution? (The true solution is $x(t) = e^{-1000t}(9979/9999) + e^{-t/10}(20/9999)$, $y(t) = 2e^{-t/10}$.) Turn in a plot of $x(t)$ and $y(t)$ that shows what happens if you choose the time step too large. Also turn in a plot of $x(t)$ and $y(t)$ once you have found a good size time step.

Try solving this system of ODE's with the trapezoidal method. Now what size time step do you need to obtain a reasonably accurate approximate solution. Can you explain why the second-order trapezoidal method is better than the fourth-order Runge-Kutta method for this problem?