Markov Chains

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A Markov chain is a special kind of stochastic process, in which a system can be in a finite number of states and the probabilities governing changes between the states do not depend on the history of the system. That is, knowing the current state of the system we can give the probabilities of future states, without regard to how the system reached its current state.

1 Examples

- Drunkard's walk. A drunkard starts at position x below, and moves to the right one space with probability .5 and to the left one space with probability .5. If he reaches the bar, he stays there, and if he reaches home, he stays there. What is the probability $p(x)$ that he reaches home before reaching the bar?

  Observe that $p(0) = 0$ and $p(5) = 1$, since the drunk does not leave either the bar or his home. For $x = 1, 2, 3, 4$, $p(x) = .5 * p(x-1) + .5 * p(x+1)$, since he moves left with probability .5 and right with probability .5.

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One could solve this problem by Monte Carlo simulation. Start, say, 10000 drunks at position x and determine their movements via a random number generator. For example, one could use a uniform random number generator such as `rand` and move left if the number is less than .5, right if it is greater than .5. Count the fraction of drunks that reach home before they reach the bar. This is not the most efficient way to solve the problem, however.

We can think of this problem as having six states, 0 through 5, corresponding to the current position of our walker. Define the *transition matrix* P so that

$$P_{ij} = \text{probability of going from state } j \text{ to state } i.$$

[Note: Sometimes the transition matrix is defined as the *transpose* of the one I have given here. Then P_{ij} is the probability of going from state i to state j. To use this definition, one must apply row vectors to P from the left, whereas I will put P on the left and apply it to column vectors on the right.]

The transition matrix for this problem is:

$$P = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{pmatrix}.$$

Note that the entries in a transition matrix are all greater than or equal to zero and the sum of the entries in any column is one: $\sum_i P_{ij} = 1 \ \forall j$. Such a matrix may also be called a *probability matrix*.

Suppose the walker starts at position $x = 2$. The initial state is described by the vector

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

To see the possible positions and their probabilities after one step, apply the transition matrix P to $x^{(0)}$ to obtain

$$x^{(1)} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{pmatrix},$$

indicating that with probability 1/2 the walker is in position 1, while with probability 1/2 he is in position 3. Continuing in this way, we wish to compute $(\lim_{n\to\infty} P^n)x^{(0)}$. All entries of this vector except the first and the last will be 0, since the walker will eventually end up either at home or at the bar. (These are called *absorbing states*.) The last entry of this vector is the desired answer $p(x)$.

We could do this computation with MATLAB by entering the matrix P and the vector $x^{(0)}$ and applying P to $x^{(0)}$ many, many times until a pattern starts to emerge. Try it if you are curious.

Another approach is the following. We could compute the *eigendecomposition* of P: $P = V\Lambda V^{-1}$, where Λ is the diagonal matrix of eigenvalues, and V is the matrix of eigenvectors. Then $P^n = V\Lambda^n V^{-1}$ and $\lim_{n\to\infty} P^n = V(\lim_{n\to\infty} \Lambda^n)V^{-1}$. It is easy to compute $\lim_{n\to\infty} \Lambda^n$, since a diagonal matrix raised to a power just consists of the individual diagonal entries raised to that power. The eigenvalues and eigenvectors of our matrix P are known, but we could also compute (approximations to) them using MATLAB. Entering the matrix P and typing `[V,Lam] = eig(P)`, MATLAB returns

```
V =

    1.0000         0    0.5721   -0.1017    0.3707   -0.2185
         0         0   -0.2185    0.3679   -0.5122    0.5721
         0         0   -0.3536   -0.5952   -0.3166   -0.3536
         0         0   -0.3536    0.5952    0.3166   -0.3536
         0         0   -0.2185   -0.3679    0.5122    0.5721
         0    1.0000    0.5721    0.1017   -0.3707   -0.2185


Lam =

    1.0000         0         0         0         0         0
         0    1.0000         0         0         0         0
         0         0    0.8090         0         0         0
         0         0         0   -0.8090         0         0
         0         0         0         0    0.3090         0
         0         0         0         0         0   -0.3090
```

and the inverse of `V` can be computed by typing `Vinv = inv(V)`

```
Vinv =

    1.0000    0.8000    0.6000    0.4000    0.2000         0
         0    0.2000    0.4000    0.6000    0.8000    1.0000
         0   -0.6325   -1.0233   -1.0233   -0.6325         0
         0    0.3757   -0.6078    0.6078   -0.3757         0
         0   -0.7063   -0.4365    0.4365    0.7063         0
         0    0.6325   -0.3909   -0.3909    0.6325         0
```

The matrix P has two eigenvalues that are equal to one and the others are less than one. When these are raised to high powers they will approach zero. Thus

$$\lim_{n\to\infty} \Lambda^n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 1 & .8 & .6 & .4 & .2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .2 & .4 & .6 & .8 & 1 \end{pmatrix}.$$

From this expression it is easy to determine the value of $p(x)$ for any x; it is the corresponding entry of the last row. The probability $p(x)$ of reaching home before reaching the bar is $x/5$, and the probability of reaching the bar first is $1 - x/5$.

An obvious generalization of this problem is to assign different probabilities for moving right and moving left, say, the walker moves right with probability $3/5$ and left with probability $2/5$. Now determine the probability $p(x)$ of reaching home before reaching the bar. The same techniques can be applied.

- Electrical Networks. The "silly" sounding problem above actually has application in many areas. Consider an electrical network with equal resistors in series and a unit voltage across the ends.

Voltages $v(x)$ will be established at points $x = 0, 1, 2, 3, 4, 5$. We have grounded the point $x = 0$ so that $v(0) = 0$, and there is no resistor between the source and point $x = 5$, so that $v(5) = 1$. By Kirchoff's laws, the current flowing into x must be equal to the current flowing out. By Ohm's Law, if points x and y are separated by a resistor of strength R, then the current i_{xy} that flows from x to y is

$$i_{xy} = \frac{v(x) - v(y)}{R}.$$

Thus for $x = 1, 2, 3, 4$, we have

$$\frac{v(x-1) - v(x)}{R} = \frac{v(x) - v(x+1)}{R}.$$

Multiplying by R and combining like terms we see that $v(x) = .5 * v(x-1) + .5 * v(x+1)$. This is exactly the same formula (with the same boundary conditions) that we found for $p(x)$ in the previous problem.

Suppose the resistors are not of equal strength, say, the resistor between points x and $x+1$ has strength $R_{x+1/2}$. Then

$$\frac{v(x-1) - v(x)}{R_{x-1/2}} = \frac{v(x) - v(x+1)}{R_{x+1/2}},$$

so

$$v(x) = v(x-1)\frac{R_{x+1/2}}{R_{x-1/2} + R_{x+1/2}} + v(x+1)\frac{R_{x-1/2}}{R_{x-1/2} + R_{x+1/2}}.$$

This is the same as the problem in which the walker goes right with probability $R_{x-1/2}/(R_{x-1/2}+R_{x+1/2})$ and left with probability $R_{x+1/2}/(R_{x-1/2}+R_{x+1/2})$.

- Winning a Game. The following is still another problem analogous to the previous two. Jack and Jill play rummy for \$1 a hand. Jack starts with \$2 and Jill starts with \$3. There are six possible states: Jack has no money and Jill has \$5, Jack has \$1 and Jill has \$4, etc. The game continues until one of the players has no money. If the players are equally skilled, then the probability of winning a hand is .5 for each. If we ask, what is the probability that Jack wins the game then it is the same as the probability that the drunk of problem 1, starting at position $x = 2$, makes it home.