Line Transversals in the Plane

Shira Zerbib

Iowa State University

Joint with Daniel McGinnis

A family of sets can be can be pierced by k lines if there are k lines whose union intersects every set in the family.

A family of sets can be can be pierced by k lines if there are k lines whose union intersects every set in the family.

A family has a line transversal if it can be pierced by one line.

A family of sets can be can be pierced by k lines if there are k lines whose union intersects every set in the family.

A family has a line transversal if it can be pierced by one line.

A family of compact convex sets F in $\mathbb{R}^2$ has the $T(k)$ property if every k sets in F have a line transversals.

A family of sets can be can be pierced by k lines if there are k lines whose union intersects every set in the family.

A family has a line transversal if it can be pierced by one line.

A family of compact convex sets F in $\mathbb{R}^2$ has the $T(k)$ property if every k sets in F have a line transversals.

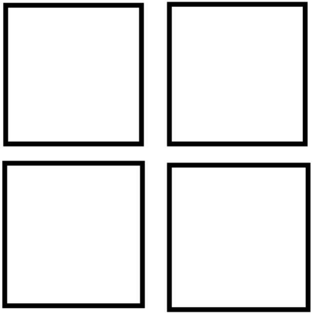
Example: A family with the $T(3)$ property

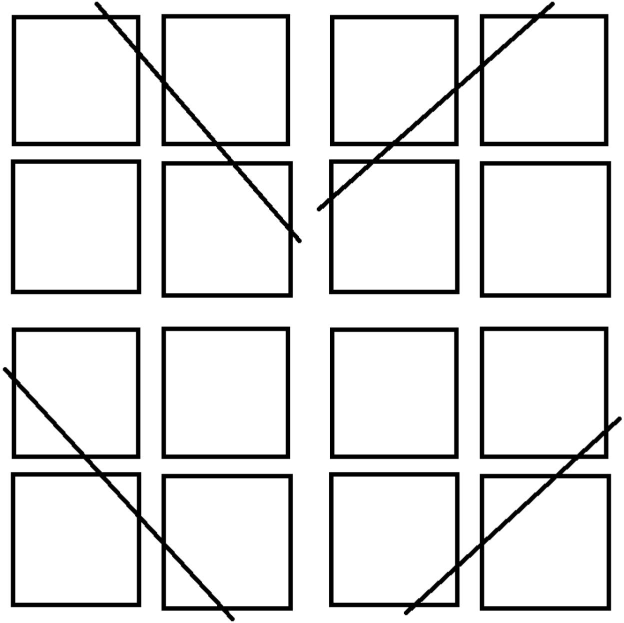
A family of sets can be can be pierced by k lines if there are k lines whose union intersects every set in the family.

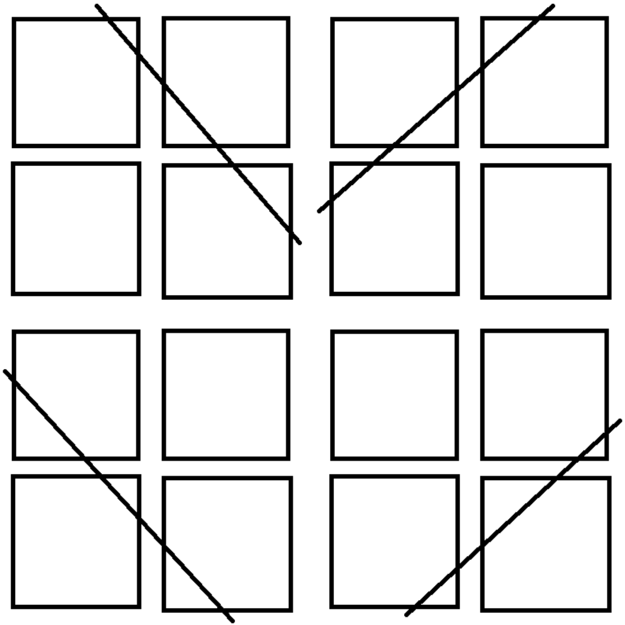
A family has a line transversal if it can be pierced by one line.

A family of compact convex sets F in $\mathbb{R}^2$ has the $T(k)$ property if every k sets in F have a line transversals.

Example: A family with the $T(3)$ property







$T(K)$ property $\implies$ $T(K-1)$ property.

How many lines are needed to pierce a family with the $T(k)$ property?

How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Rightarrow$ pierced by one line.

How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Longrightarrow$ pierced by one line.

Eckhoff (1964): $T(4)$ property $\Longrightarrow$ pierced by 2 lines.

How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Longrightarrow$ pierced by one line.

Eckhoff (1964): $T(4)$ property $\Longrightarrow$ pierced by 2 lines.

This shows: for any $k \geq 4$, $T(k)$ property $\Longrightarrow$ pierced by 2 lines.

How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Longrightarrow$ pierced by one line.

Eckhoff (1964): $T(4)$ property $\Longrightarrow$ pierced by 2 lines.

This shows: for any $k \geq 4$, $T(k)$ property $\Longrightarrow$ pierced by 2 lines.

$T(2)$ property $\not\Longrightarrow$ pierced by a constant number of line (Example: a set of point in general position).

How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Longrightarrow$ pierced by one line.

Eckhoff (1964): $T(4)$ property $\Longrightarrow$ pierced by 2 lines.

This shows: for any $k \geq 4$, $T(k)$ property $\Longrightarrow$ pierced by 2 lines.

$T(2)$ property $\not\Longrightarrow$ pierced by a constant number of line (Example: a set of point in general position).

What about $k = 3$?

How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Longrightarrow$ pierced by one line.

Eckhoff (1964): $T(4)$ property $\Longrightarrow$ pierced by 2 lines.

This shows: for any $k \geq 4$, $T(k)$ property $\Longrightarrow$ pierced by 2 lines.

$T(2)$ property $\not\Longrightarrow$ pierced by a constant number of line (Example: a set of point in general position).

What about $k = 3$?

Kramer (1975): $T(3)$ property $\Longrightarrow$ pierced by 5 lines.

How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Longrightarrow$ pierced by one line.

Eckhoff (1964): $T(4)$ property $\Longrightarrow$ pierced by 2 lines.

This shows: for any $k \geq 4$, $T(k)$ property $\Longrightarrow$ pierced by 2 lines.

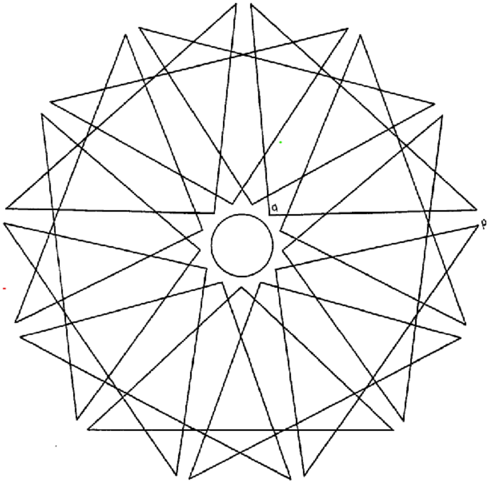
$T(2)$ property $\not\Longrightarrow$ pierced by a constant number of line (Example: a set of point in general position).

What about $k = 3$?

Kramer (1975): $T(3)$ property $\Longrightarrow$ pierced by 5 lines.

Eckhoff (1974): $T(3)$ property $\not\Longrightarrow$ pierced by 2 lines.

Eckhoff (1974): $T(3)$ property $\not\Rightarrow$ pierced by 2 lines.



How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Longrightarrow$ pierced by one line.

Eckhoff (1964): $T(4)$ property $\Longrightarrow$ pierced by 2 lines.

This shows: for any $k \geq 4$, $T(k)$ property $\Longrightarrow$ pierced by 2 lines.

$T(2)$ property $\not\Longrightarrow$ pierced by a constant number of line (Example: a set of point in general position).

What about $k = 3$?

Kramer (1975): $T(3)$ property $\Longrightarrow$ pierced by 5 lines.

Eckhoff (1974): $T(3)$ property $\not\Longrightarrow$ pierced by 2 lines.

How many lines are needed to pierce a family with the $T(k)$ property?

Santalo (1940): For any k, $T(k)$ property $\not\Longrightarrow$ pierced by one line.

Eckhoff (1964): $T(4)$ property $\Longrightarrow$ pierced by 2 lines.

This shows: for any $k \geq 4$, $T(k)$ property $\Longrightarrow$ pierced by 2 lines.

$T(2)$ property $\not\Longrightarrow$ pierced by a constant number of line (Example: a set of point in general position).

What about $k = 3$?

Kramer (1975): $T(3)$ property $\Longrightarrow$ pierced by 5 lines.

Eckhoff (1974): $T(3)$ property $\not\Longrightarrow$ pierced by 2 lines.

Eckhoff (1993): $T(3)$ property $\Longrightarrow$ pierced by 4 lines.

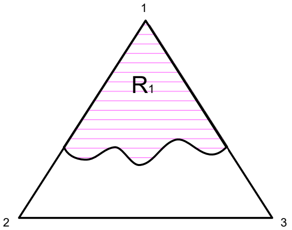
Question (Eckhoff, 1993)

Is it true that $T(3)$ property $\Longrightarrow$ pierced by 3 lines?

Theorem (McGinnis-Z., 2021+)

$T(3)$ *property* $\implies$ *pierced by 3 lines.*

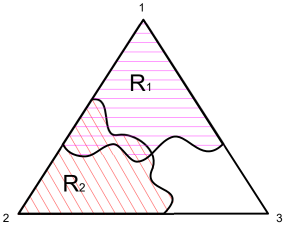
The KKM Theorem (Knaster-Kuratowski-Mazurkiewicz, 1928):



Let $R_1, \ldots, R_n$ be open sets covering

$$\Delta^{n-1} = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1\},$$

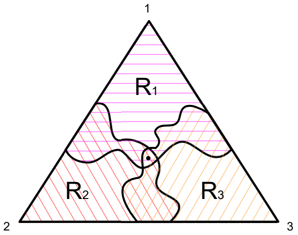
The KKM Theorem (Knaster-Kuratowski-Mazurkiewicz, 1928):



Let $R_1, \ldots, R_n$ be open sets covering

$$\Delta^{n-1} = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1\},$$

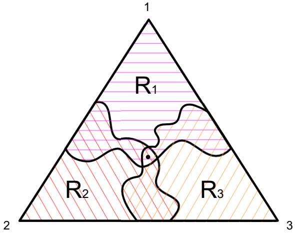
The KKM Theorem (Knaster-Kuratowski-Mazurkiewicz, 1928):



Let $R_1, \ldots, R_{n-1}$ be open sets covering

$$\Delta_{n-1} = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1\},$$

The KKM Theorem (Knaster-Kuratowski-Mazurkiewicz, 1928):

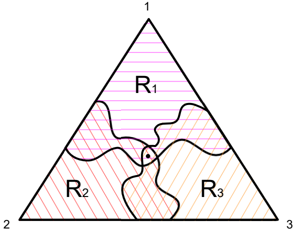


Let $R_1, \ldots, R_{n-1}$ be open sets covering

$$\Delta_{n-1} = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1\},$$

such that for every face σ, $\sigma \subseteq \bigcup_{i \in \sigma} R_i$.

The KKM Theorem (Knaster-Kuratowski-Mazurkiewicz, 1928):



Let $R_1, \ldots, R_{n-1}$ be open sets covering

$$\Delta_{n-1} = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1\},$$

such that for every face σ, $\sigma \subseteq \bigcup_{i \in \sigma} R_i$.

Then $\bigcap_{i=1}^{n} R_i \neq \emptyset$.

Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof.

Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.

Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

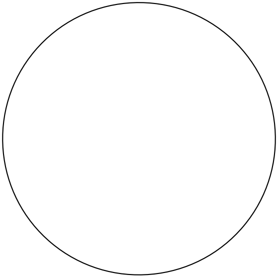
Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

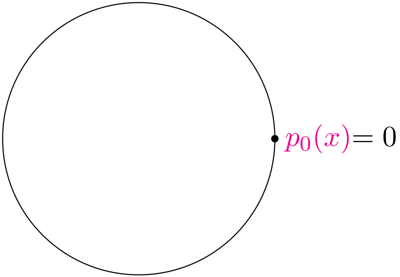


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

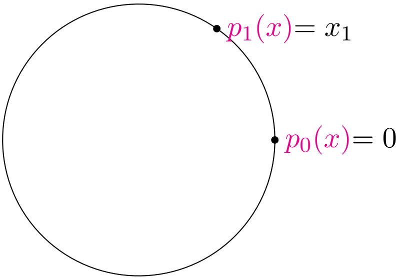


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

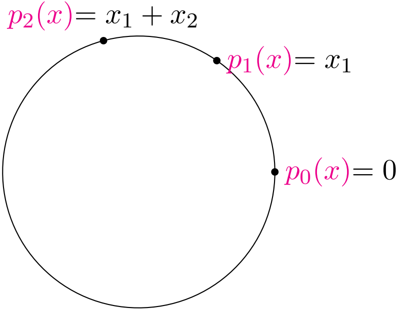


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

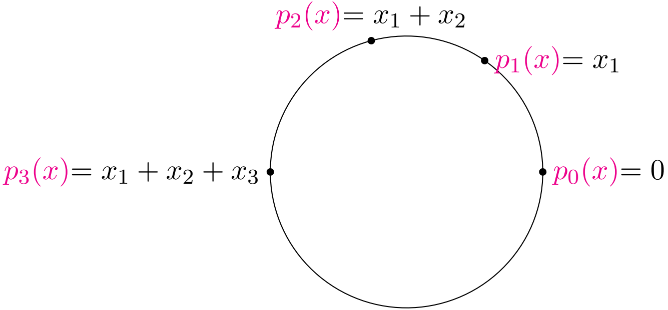


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

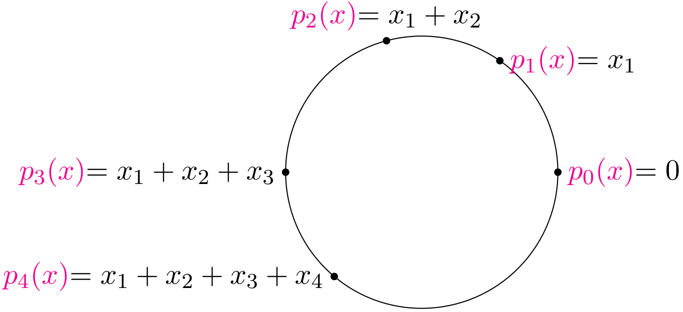


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

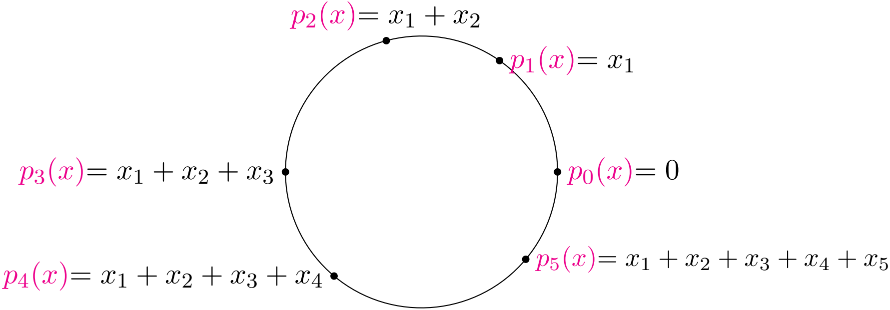


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

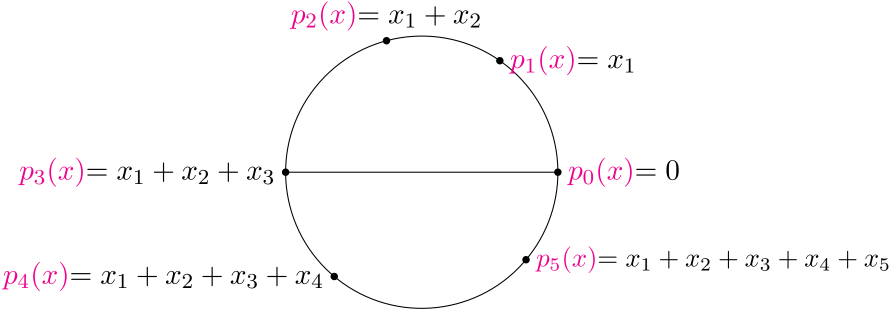


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

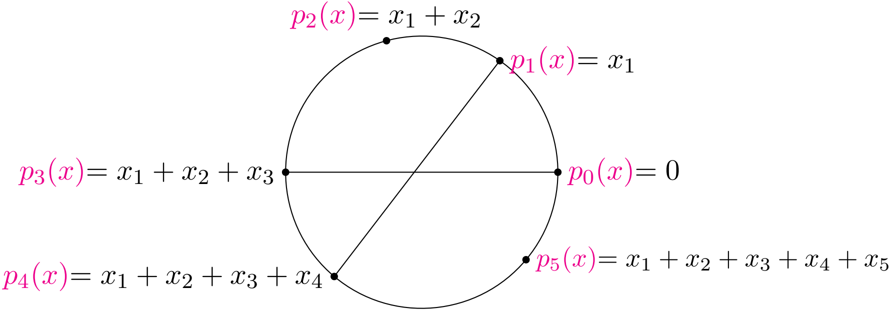


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

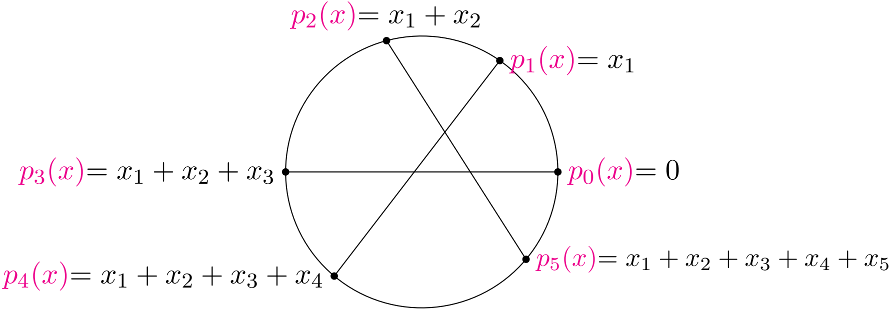


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

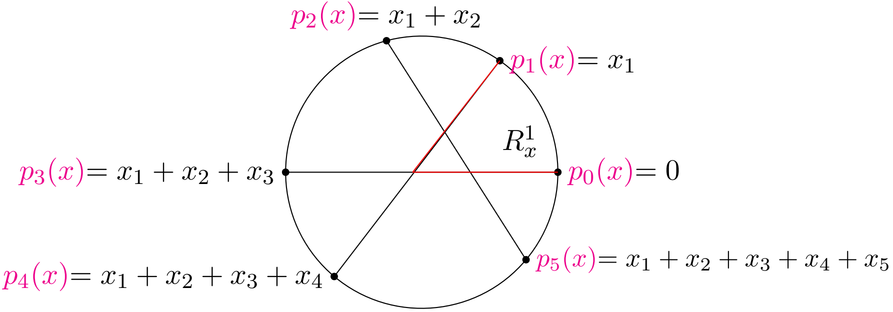


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

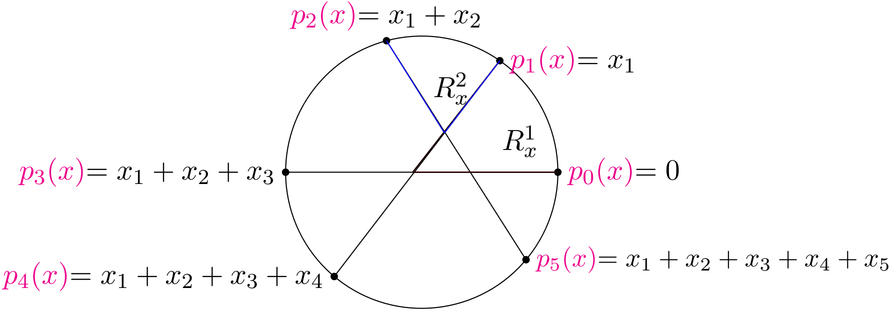


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

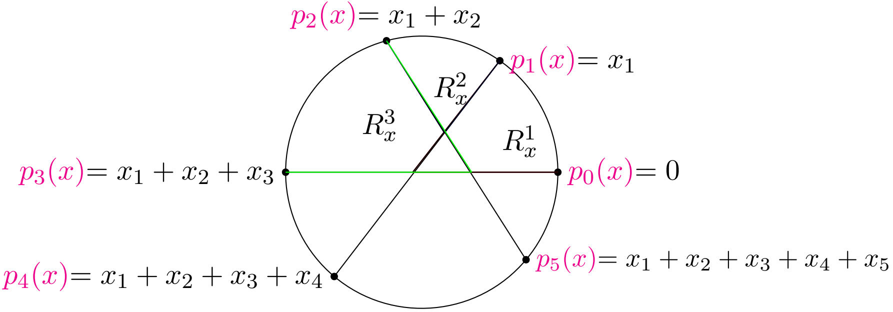


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

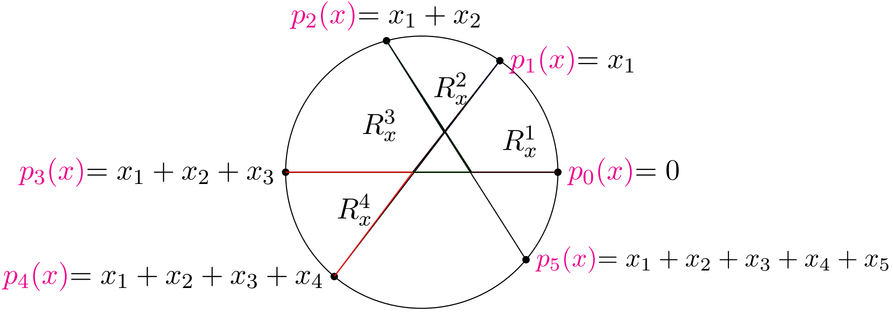


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

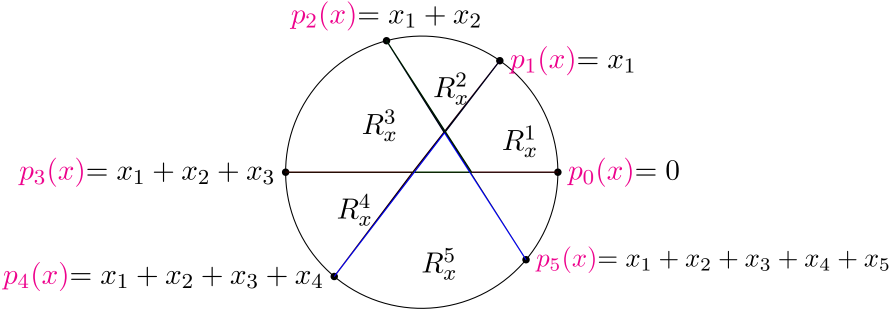


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.

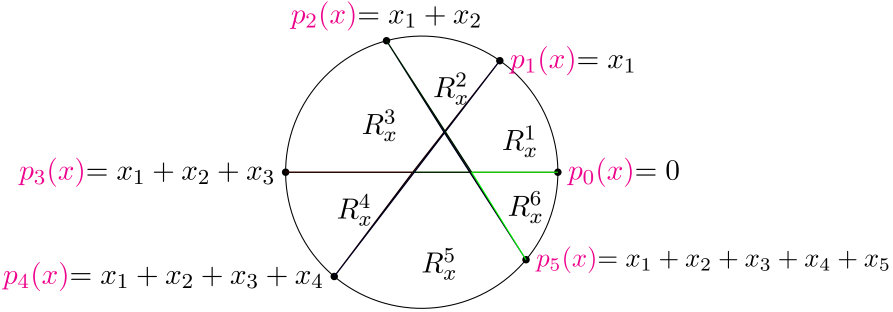


Theorem (McGinnis-Z.): $T(3)$ property $\implies$ pierced by 3 lines.

Proof. Let F be a family of sets with the $T(3)$ property.

Scale the plane so that all the sets in F lie in a disk D of perimeter 1.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \Delta^5$.
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, $x_i \geq 0$.



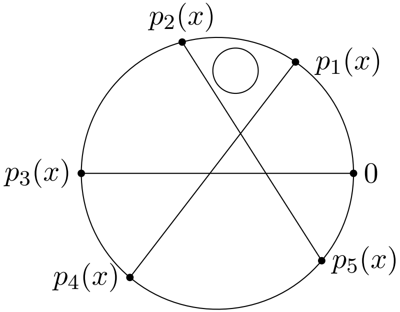
We want to define a KKM cover of Δ^5. For this, we define sets A_i covering Δ^6 as follows:

We want to define a KKM cover of Δ^5. For this, we define sets A_i covering Δ^6 as follows:

set $x \in A_i$ if R_x^i contains a set in F.

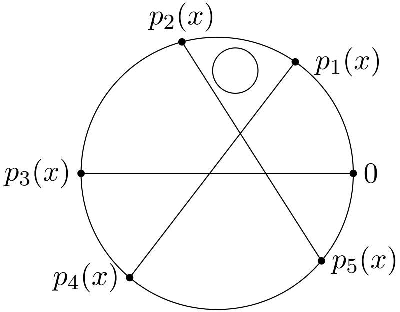
We want to define a KKM cover of Δ^5. For this, we define sets A_i covering Δ^6 as follows:

set $x \in A_i$ if R_x^i contains a set in F.



We want to define a KKM cover of Δ^5. For this, we define sets A_i covering Δ^6 as follows:

set $x \in A_i$ if R_x^i contains a set in F.



Here: $x \in A_2$

- If there is some $x \notin \cup A_i$, then we are done:
no set lies in a region R_x^i, so all the sets are pierced by the three lines

$$\overline{p_0(x)p_3(x)},\ \overline{p_1(x)p_4(x)},\ \overline{p_2(x)p_5(x)}$$

- If there is some $x \notin \cup A_i$, then we are done:
  no set lies in a region R_x^i, so all the sets are pierced by the three lines

$$\overline{p_0(x)p_3(x)},\ \overline{p_1(x)p_4(x)},\ \overline{p_2(x)p_5(x)}$$

- So we may assume $\Delta^5 \subset \bigcup A_i$

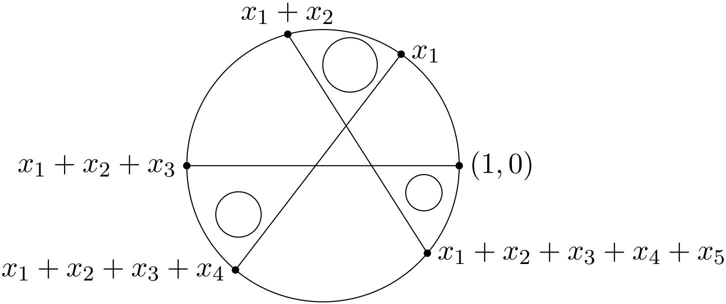
- If there is some $x \notin \cup A_i$, then we are done:
  no set lies in a region R_x^i, so all the sets are pierced by the three lines

$$\overline{p_0(x)p_3(x)},\ \overline{p_1(x)p_4(x)},\ \overline{p_2(x)p_5(x)}$$

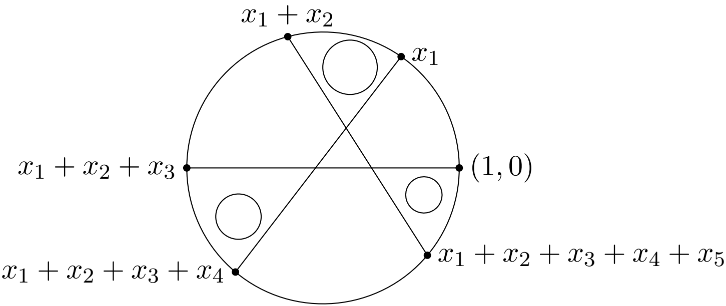
- So we may assume $\Delta^5 \subset \bigcup A_i$
- Claim: In this case, $A_1, \ldots, A_6$ form a KKM cover of Δ^5.

- By the KKM theorem there exists some $x \in \bigcap_{i=1}^{6} A_i$.

- By the KKM theorem there exists some $x \in \bigcap_{i=1}^{6} A_i$.

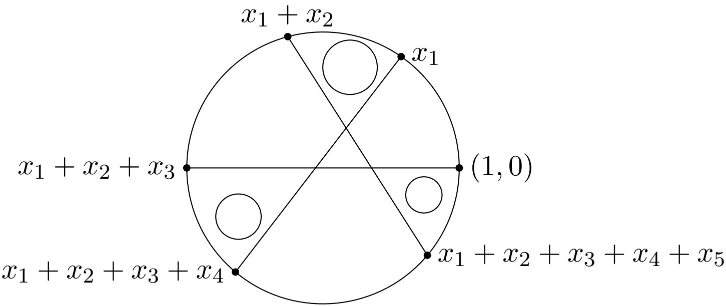


- By the KKM theorem there exists some $x \in \bigcap_{i=1}^{6} A_i$.



- $\implies$ There are 3 sets in F that are not pierced by a line

- By the KKM theorem there exists some $x \in \bigcap_{i=1}^{6} A_i$.



- $\implies$ There are 3 sets in F that are not pierced by a line
- a contradiction to the $T(3)$ property. $\square$

Colorful Versions

Theorem (McGinnis – Z. 2021+)

Let $F_1, \ldots, F_6$ be six families of compact convex sets in $\mathbb{R}^2$.
If every $A \in F_i, B \in F_j, C \in F_k$, $i < j < k$, have a line transversal,
then there exists $i \in [6]$ such that F_i is pierced by 3 lines.

Colorful Versions

Theorem (McGinnis – Z. 2021+)

Let $F_1, \dots, F_6$ be six families of compact convex sets in $\mathbb{R}^2$.
If every $A \in F_i, B \in F_j, C \in F_k$, $i < j < k$, have a line transversal,
then there exists $i \in [6]$ such that F_i is pierced by 3 lines.

If all the F_i are the same we get the previous theorem.

Colorful Versions

Theorem (McGinnis – Z. 2021+)

Let $F_1, \ldots, F_6$ be six families of compact convex sets in $\mathbb{R}^2$.
If every $A \in F_i, B \in F_j, C \in F_k$, $i < j < k$, have a line transversal,
then there exists $i \in [6]$ such that F_i is pierced by 3 lines.

If all the F_i are the same we get the previous theorem.

Theorem (McGinnis – Z. 2021+)

Let $F_1, \ldots, F_4$ be four families of compact convex sets in $\mathbb{R}^2$.
If any collection of four sets, one from each F_i, has a line transversal,
then there exists $i \in [4]$ such that F_i is pierced by 2 lines.

Colorful Versions

Theorem (McGinnis – Z. 2021+)

Let $F_1, \ldots, F_6$ be six families of compact convex sets in $\mathbb{R}^2$. If every $A \in F_i, B \in F_j, C \in F_k$, $i < j < k$, have a line transversal, then there exists $i \in [6]$ such that F_i is pierced by 3 lines.

If all the F_i are the same we get the previous theorem.

Theorem (McGinnis – Z. 2021+)

Let $F_1, \ldots, F_4$ be four families of compact convex sets in $\mathbb{R}^2$. If any collection of four sets, one from each F_i, has a line transversal, then there exists $i \in [4]$ such that F_i is pierced by 2 lines.

If all the F_i are the same we get

Eckhoff (1964): $T(4)$ property $\implies$ pierced by 2 lines.

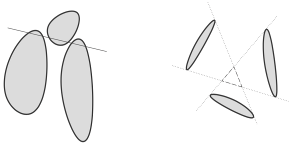
Proof.

Use the colorful KKM theorem (Gale, 1982).

Weakening the $T(3)$ condition

Definition (Holmsen 2013): Three sets A, B, C form a tight triple if

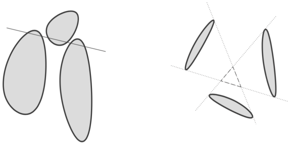
$$\text{conv}(A \cup B) \cap \text{conv}(A \cup C) \cap \text{conv}(B \cup C) \neq \emptyset.$$



Weakening the $T(3)$ condition

Definition (Holmsen 2013): Three sets A, B, C form a tight triple if

$$\mathrm{conv}(A \cup B) \cap \mathrm{conv}(A \cup C) \cap \mathrm{conv}(B \cup C) \neq \emptyset.$$

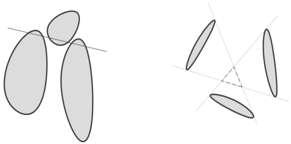


Note: A, B, C have a line transversal $\implies$ A, B, C are a tight triple.

Weakening the $T(3)$ condition

Definition (Holmsen 2013): Three sets A, B, C form a tight triple if

$$\text{conv}(A \cup B) \cap \text{conv}(A \cup C) \cap \text{conv}(B \cup C) \neq \emptyset.$$



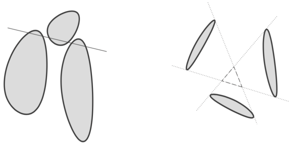
Note: A, B, C have a line transversal $\implies$ A, B, C are a tight triple.

Definition: A family of sets F has the TT property if every three sets in F form a tight triple.

Weakening the $T(3)$ condition

Definition (Holmsen 2013): Three sets A, B, C form a tight triple if

$$\text{conv}(A \cup B) \cap \text{conv}(A \cup C) \cap \text{conv}(B \cup C) \neq \emptyset.$$



Note: A, B, C have a line transversal $\implies$ A, B, C are a tight triple.

Definition: A family of sets F has the TT property if every three sets in F form a tight triple.

Note: $T(3)$ property $\implies$ TT property.

Weakening the $T(3)$ condition

Theorem (Holmsen, 2013)

TT property $\implies$ there is a line intersecting at least $\frac{1}{8}|F|$ members of F.

Weakening the $T(3)$ condition

Theorem (Holmsen, 2013)

TT property $\implies$ there is a line intersecting at least $\frac{1}{8}|F|$ members of F.

Theorem (McGinnis – Z., 2021+)

TT property $\implies$ pierced by 3 lines.

Weakening the $T(3)$ condition

Theorem (Holmsen, 2013)

TT property $\implies$ there is a line intersecting at least $\frac{1}{8}|F|$ members of F.

Theorem (McGinnis – Z., 2021+)

TT property $\implies$ pierced by 3 lines.

Theorem (McGinnis – Z., 2021+)

TT property $\implies$ there is a line intersecting at least $\frac{1}{3}|F|$ members of F.

Open Problems

Conjecture (Martínez – Roldán – Rubin, 2020)

There exists a constant c with the following property:
Suppose that F is an intersecting family of compact convex in $\mathbb{R}^3$.
Then there a line intersecting $c|F|$ members of F.

Open Problems

Conjecture (Martínez – Roldán – Rubin, 2020)

There exists a constant c with the following property:
Suppose that F is an intersecting family of compact convex in $\mathbb{R}^3$.
Then there a line intersecting $c|F|$ members of F.

Bárány (2021): true for cylinders.

Fractional Versions

Question

What is the largest constant $0 < \alpha(k) < 1$ such that for any family F with the $T(k)$ property, there is a line intersecting $\alpha(k)|F|$ members of F?

- $\alpha(k) \longrightarrow 1$ as $k \longrightarrow \infty$ (Katchalski, Liu 1980).
- $\alpha(k) \leq \frac{k-2}{k-1}$ (Holmsen 2010)
- $\frac{1}{3} \leq \alpha(3) \leq \frac{1}{2}$
- $\frac{1}{2} \leq \alpha(4) \leq \frac{2}{3}$

Open: What are $\alpha(3)$ and $\alpha(4)$?

Thank You!