Topological entropy on Hubbard trees

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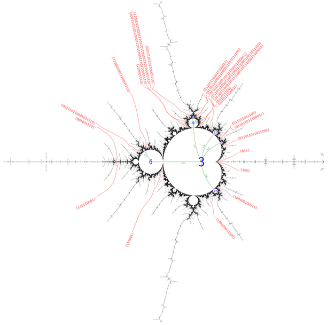
- Consider quadratic map $f_c = z \mapsto z^2 + c$. The set of points z that does not go to infinity is called the **filled Julia set**. The set of parameters c where the filled Julia set is connected (or $f_c^n(0)$ is bounded as $n \to \infty$) is called the **Mandelbrot set**.
- If c satisfies that $\{f_c^n(0)\}$ is finite, we call it a **post critically finite** parameter. If $f_c^n(0) = 0$ for some $n > 0$, we call it **superattracting**. These are the centers of hyperbolic components. The other post critical points are called **Misiurewicz**.



- External rays on Mandelbrot set are image of rays $\{te^{2\pi i\theta} : t > 1\}$ under uniformization map to the complement of the Mandelbrot set.
- Rays with angle in $\mathbb{Q}\pi$ lands on Misiurewicz points or on bases of a hyperbolic component, hence correspond to PCF maps.
- When the map is PCF, there is a unique way to connect the points in critical orbit within the filled Julia set, such that it is preserved by the map f_c. This is called the Hubbard tree, and the topological entropy of f_c on it is called the **core entropy**.
- In the special case of $c \in \mathbb{R}$, the Hubbard tree is a line segment and study of dynamics on Hubbard tree is the same as the study of unimodal interval maps.
- The calculation of core entropy is as follows: the forward orbit of the critical point, as well as the branched points of the tree, cut it into finitely many segments which form a Markov decomposition. The exp of the core entropy is the leading eigenvalue of the resulting incidence matrix. Let Λ_c be the set of all eigenvalues of this Markov incidence matrix.

Statement of results

- **Theorem A:** When c is a superattracting parameter, $\Lambda_c \cup \{z : |z| \leq 1\}$ moves continuously with the external angle.
- Previously Tiozzo proved that the leading eigenvalue moves continuously, and this Theorem gives a stronger result in superattracting case.
- For any q, there are some c at the tip of Mandelbrot set whose Hubbard tree is star shaped with q branches, and $f_c^q(0)$ is a fixed point at the end of one of the branches. The arc between such a c to the main cardioid is called a **principal vein**.
- **Theorem B:** For any two superattracting parameters c and c' on a principal vein, if c' is further from the main cardioid than c, then for any ϵ', any $z \in \Lambda_c$, $|z| < 1$, there is another superattracting parameter on the same vein c'', whose entropy is ϵ-close to c' and some point in $\Lambda_{c''}$ is ϵ-close to z.
- The case for $1/2$ principal vein was previously proved by Bray-Davis-Lindsey-W.

Applications and further questions

- Theorem B gives an algorithm to test whether or not a point is in the "Thurston Teapot" of a principal vein, defined as:

$$T_q = \overline{\{(z, e^{h(f_c)}) : z \in \Lambda_c, c \text{ superattracting on the vein}\}}$$

  Therefore it provides a necessary condition for a number to be core entropy of superattracting parameters on a vein.
- The technique might be generalizable to higher degree, or to train track maps or even general graph maps.

Ideas for Theorem A:

The core entropy of such PCF maps can be calculated using Thurston's algorithm as follows:

- Let $x_1 = \theta mod 1$, $x_i = 2x_{i-1} mod 1$. Let S be the finite set consisting of all x_i.
- Divide $\mathbb{R}/\mathbb{Z}$ into two segments at $\theta/2$ and $\theta/2 + 1/2$.
- Let V_θ be set of subsets of S with 2 points, M_θ be a map from $\mathbb{R}^{V_\theta}$ to itself, sending (x_i, x_j) to
    - (x_{i+1}, x_{j+1}), if x_i and x_j lie in the closure of the same segment.
    - $(x_1, x_{i+1}) + (x_1, x_{j+1})$, if otherwise. (x_1, x_1) is set as 0.

Now lift the finite directed graph to an appropriate infinite weak cover (another directed graph with a surjection to it, such that outgoing edges can be lifted but not incoming edges).

Ideas for Theorem B:

The dynamics of f_c on Hubbard tree for c in principal vein can be described as follows:

- The Hubbard tree is star shaped with q arms, one divided into I_0 and I_1 by 0, the other denoted as I_2, ..., I_q.
- I_j get sent to I_{j+1} for $j = 1, \ldots, q-1$.
- I_q gets sent to $I_0 \cup I_1$.
- I_1 is sent to a subinterval of $I_0 \cup I_1 \cup I_2$.

Now consider the itinerary of critical value (the infinite string of labels of interval its forward orbit belongs to), and the itinerary of the first return map on $I_0 \cup I_1 \cup I_2$. The two itineraries can be related with a finite state transducer, and the first return map on $I_0 \cup I_1 \cup I_2$ can be studied via Milnor-Thurston kneading theory as it is unimodal.

References

The paper is being finalized and will be on arXiv soon, if interested please email me at wuchenxi2013@gmail.com for a draft. The 1/2 vein case is completely described in a prior paper arxiv:1909.10675