Matching Complexes of Polygonal Tilings

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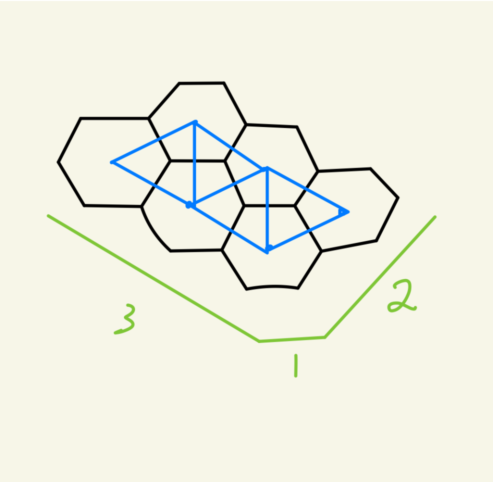
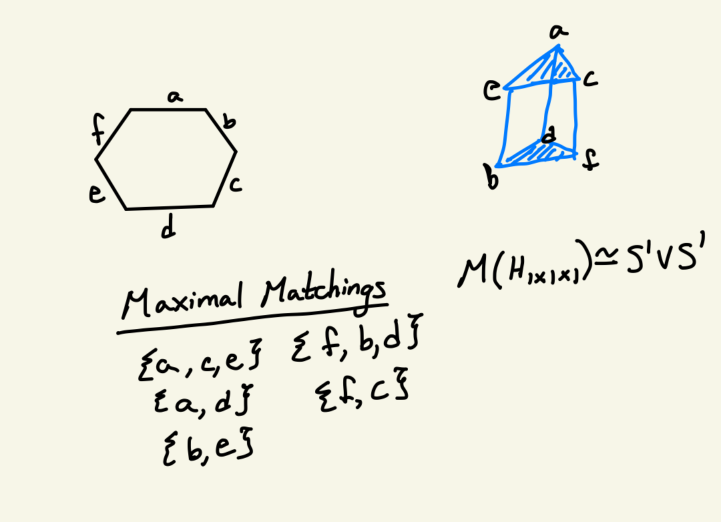
Honeycomb Graphs

A $k \times m \times n$ **honeycomb graph**, denoted $H_{k \times m \times n}$ is a hexagonal tiling whose whose congruent, opposite sides consist of k, m, and n hexagons respectively with $k, m, n \in \mathbb{Z}_{\geq 1}$



Matching Complexes

The **matching complex** of a graph is a simplicial complex whose faces are given by matchings in the graph (i.e. collections of disjoint edge sets).

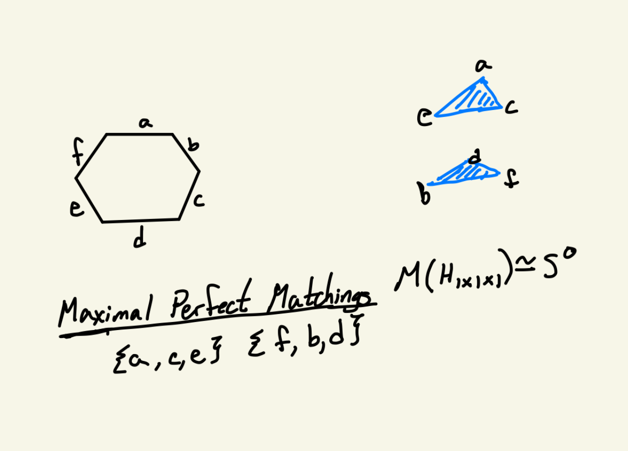


Background

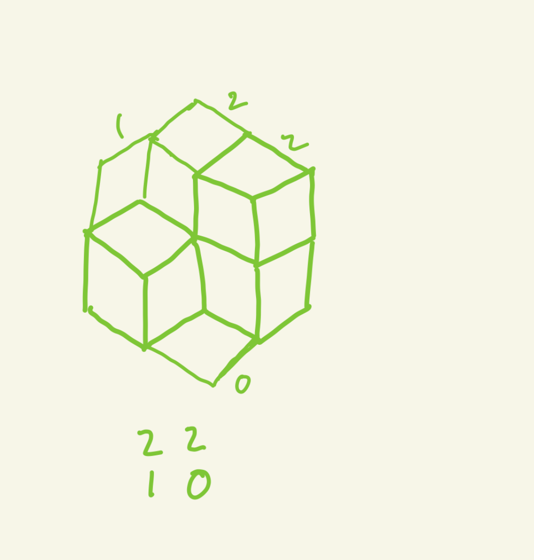
- ('05 Jonsson) In unpublished manuscript: hopotopical depth and shifted connectivity for $m \times n$ grid graphs .
- ('17 Braun and Hough) Homological bounds on $2 \times n$ grid graph
- ('19 Matsushita) Showed $n \times 2$ grid graph homotopy equivalent to wedge of spheres
- ('19 Jelić et. al)
  - Connectivity bounds for line of polygons
  - When $n \equiv 1 \pmod 3$ and $t \geq 2$, the matching complex of t $2n$-gons is homotopy equivalent to a wedge of t spheres of dimension $\frac{2nt+t}{3} - t$.
  - Connectivity bounds for $2 \times 1 \times n$ honeycomb graphs
- ('19 Matsushita) Determined the homotopy type when $n \equiv 0$ and $n \equiv 2 \pmod 3$ for lines of polygons is a wedge of spheres and showed the connectivity bounds were tight for $n \equiv 0$ but not when $n \equiv 2 \pmod 3$

Perfect Matching Complex

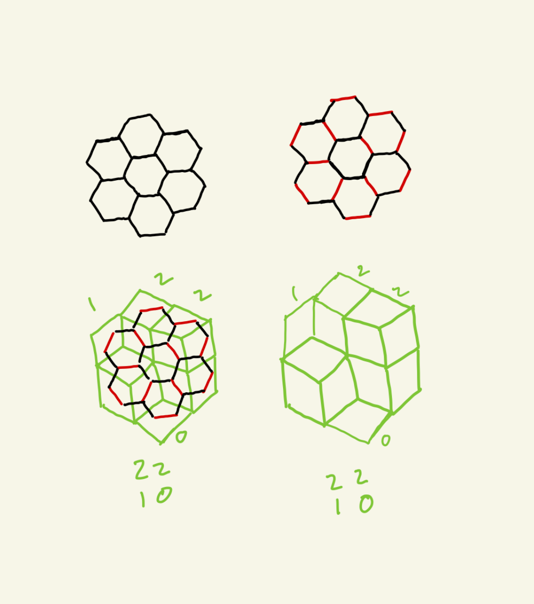
A **perfect matching** is a matching of a graph G in which every vertex of G is incident to an edge in the matching. The **perfect matching complex** of a graph G is a simplicial complex whose faces are all subsets of perfect matchings, note that this is a subcomplex of the matching complex of G.



A **plane partition** is a 2-dimensional array of integers that are non-increasing moving from left to right and top to bottom.



Plane Partitions and Perfect Matchings



Discrete Morse Matchings

- An **acyclic partial matching** in a poset P is a subset $\mu \in P \times P$ such that:
  - $(a, b) \in \mu$ implies b covers a, denoted $a = d(b)$ and $b = u(a)$
  - each $a \in P$ belongs to at most one element is μ.
  - there does not exist a cycle

$$b_1 > d(b_1) < b_2 > d(b_2) < \cdots < b_n > d(b_n) < b_1$$

    where $n \geq 2$ and all $b_i \in P$ distinct.

  Unmatched elements of an acyclic partial matching μ on P are called **critical**.

Theorem

Let Δ be a polyhedral cell complex, and μ be an acylic partial pairing on the face poset of Δ. Let c_i denote the number of critical cells of dimension i of Δ. Then Δ is homotopy equivalent to a cell complex Δ_c with c_i cells of dimension i for each $i \geq 0$, plus a single $0-$dimension cell in the case where the empty set is paired in the matching.

$1 \times 2 \times n$ honeycomb graphs

Theorem (Bayer,Jelić Multinović, V.)

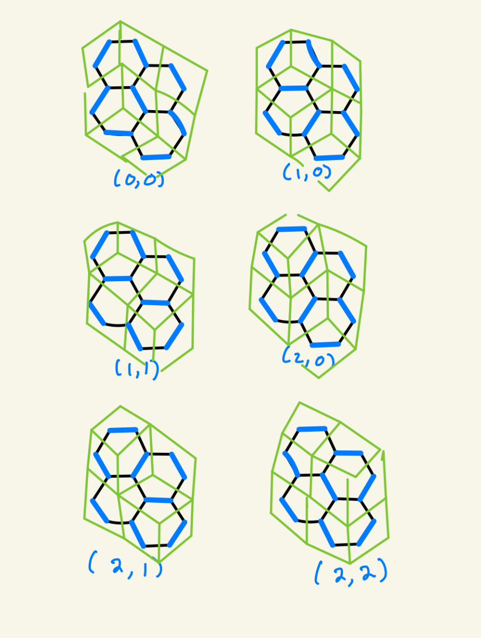
Let $H_{1\times 2\times n}$ be the honeycomb graph of dimension $1 \times 2 \times n$, $n \in \mathbb{N}$, $n \geq 2$. Then

$$\mathcal{M}_p(H_{1\times 2\times n}) \simeq S^{n-1}.$$

Theorem

Let $H_{1\times 1\times n}$ be the honeycomb graph of dimension $1 \times 1 \times n$, $n \in \mathbb{N}$, $n \geq 2$. Then $\mathcal{M}_p(H_{1\times 2\times n})$ is contractible.

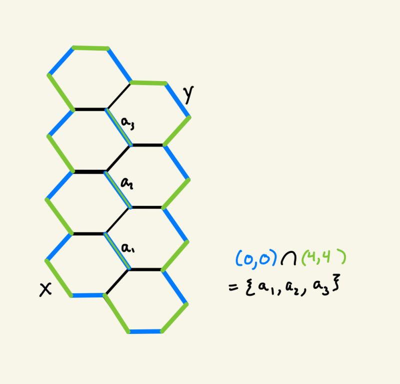
$1 \times 2 \times n$ honeycomb graphs



$1 \times 2 \times n$ honeycomb graphs

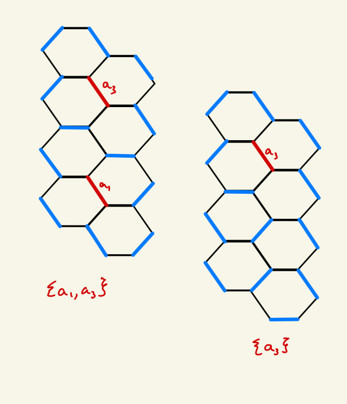
Define the discrete Morse Matching:

1. Match on x – pairs away all subposets corresponding with plane partitions not equal to $(0, 0)$.
2. Match on y – Since y is in all perfect matchings except the one corresponding to the $(n, n, \ldots, n)$ plane partition, the intersection of $(0, 0, \ldots, 0)$ and $(n, n, \ldots, n)$ along with y remains.



$1 \times 2 \times n$ honeycomb graphs

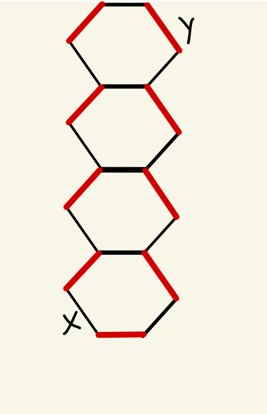
Notice that any subset of $\{a_1, a_2, a_3\}$ is contained in a maximal perfect matching.



So the only critical cell that remains is $\{a_1, a_2, a_3\} \cup \{y\}$ and for general n we have one critical cell $\{a_1, a_2, \ldots, a_{n-1}\} \cup \{y\}$.

$1 \times 1 \times n$ honeycomb graphs

In the case of $1 \times 1 \times n$ honeycomb graphs the intersection between (0) and (n) is empty.
Notice that once the height is determined all of the other edges are predetermined.
In matching on y we pair away all unmatched faces.



$1 \times m \times n$ honeycomb graphs

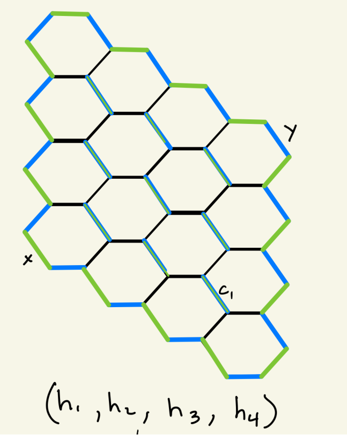
Theorem (Bayer,Jelić Multinović, V.)

Let $H_{1\times m\times n}$ be the honeycomb graph of dimension $1 \times m \times n$, $m, n \in \mathbb{N}$, and $m, n \geq 3$. Then the perfect matching complex $\mathcal{M}_p(H_{1\times m\times n})$ is contractible.

$1 \times m \times n$ honeycomb graphs

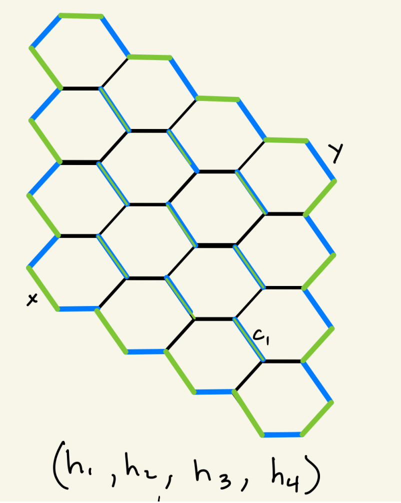
Define a discrete Morse Matching:

1. Match on x – pairs away all subposets corresponding with plane partitions not equal to $(0, 0, \ldots, 0)$.
2. Match on y – The remaining cells are contained in $((0, 0, \ldots, 0) \cap (n, n \ldots, n)) \cup \{y\}$ and contain $\{y\}$.



$1 \times m \times n$ honeycomb graphs

We make one final matching on the edge c_1.



Claim: This will pair away all unmatched cells.

$1 \times m \times n$ honeycomb graphs

- We show that $\sigma \ni c_1$ is matched if and only if $\sigma \setminus c_1$ is matched.

$1 \times m \times n$ honeycomb graphs

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- Recall that for a face τ to be previously matched that means that τ is contained in a maximal perfect matching.

$1 \times m \times n$ honeycomb graphs

- We show that $\sigma \ni c_1$ is matched if and only if $\sigma \setminus c_1$ is matched.

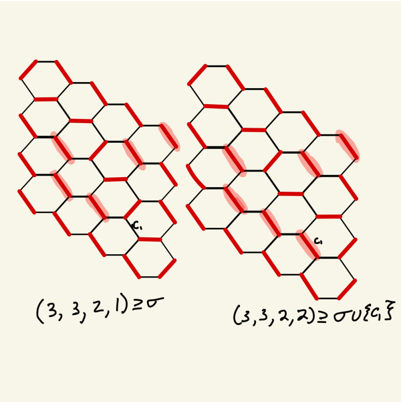
- Recall that for a face τ to be previously matched that means that τ is contained in a maximal perfect matching.

- The forward direction is clear: If $\sigma \ni c_1$ is contained in a maximal perfect matching, then $\sigma \setminus c_1$ is contained in a maximal perfect matching.

$1 \times m \times n$ honeycomb graphs

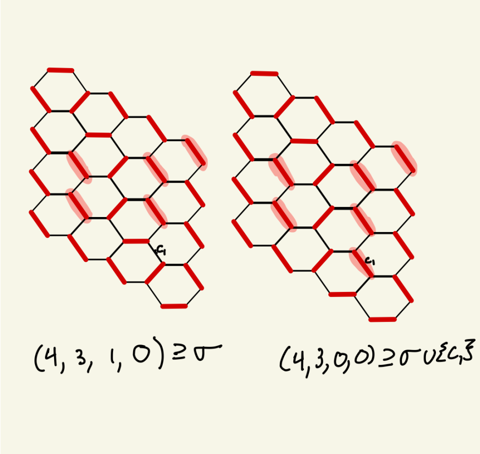
Suppose now that $\sigma \not\ni c_1$ has been previously matched. Let $(h_1, h_2, \ldots, h_m)$ denote the heights of columns $1, 2, \ldots, m$.

Case 1 $h_m \in \{0, 1\}$ and $h_{m-1} \geq 2$:



More generally, if $\sigma \subset (h_1, \ldots, h_{m-1}, 1)$ and we see $\sigma \cup \{c_1\} \subset (h_1, h_2, \ldots, 2)$.

Case 2 $h_m \in \{0,1\}$ and $h_{m-1} = 1$ and $h_{m-2} \geq 1$:



More generally, $\sigma \subset (h_1, \ldots, h_{m-2}, 1, 0)$ and we see $\sigma \cup \{c_1\} \subset (h_1, h_2, \ldots, h_{m-2}, 0, 0)$.

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