Stirling numbers for complex reflection groups

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Ordinary Stirling numbers

Complex reflection groups

Outline

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Complex reflection groups

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ρ	123	1/23, 12/3, 13/2	1/2/3
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We have $s(n, k) = (-1)^{n-k} e_{n-k}(1, 2, \ldots, n-1)$. □

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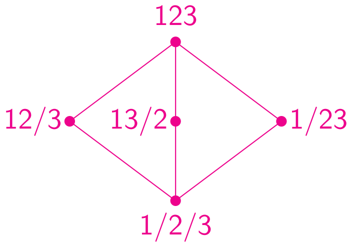

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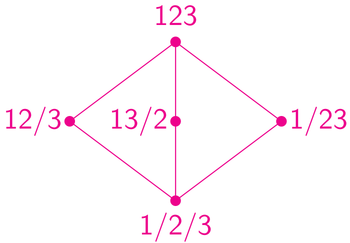

So if $\rho = B_1/\ldots/B_k \in \Pi_n$ then

$$\operatorname{rk} \rho = n - k.$$

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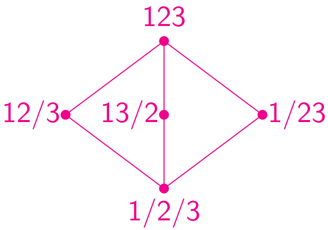

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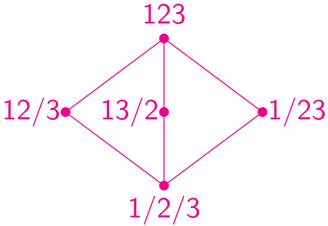

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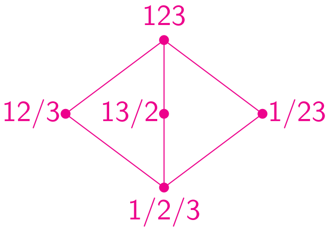

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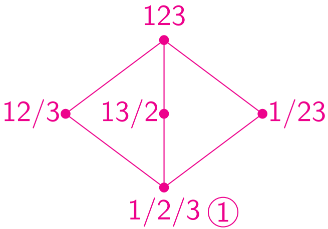

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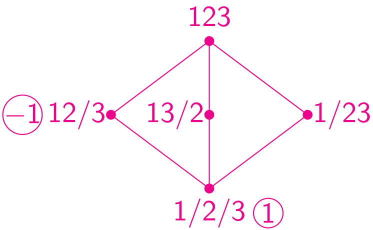

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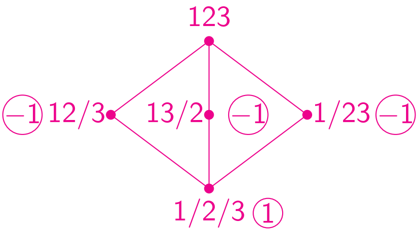

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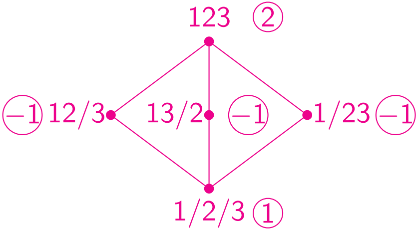

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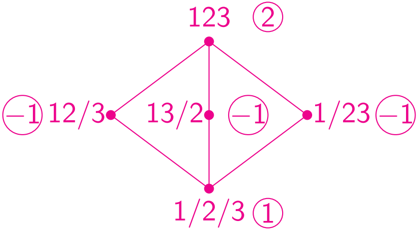

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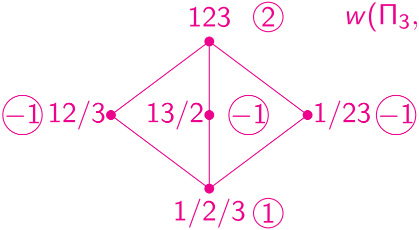

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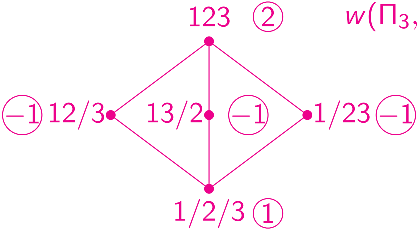

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Proposition

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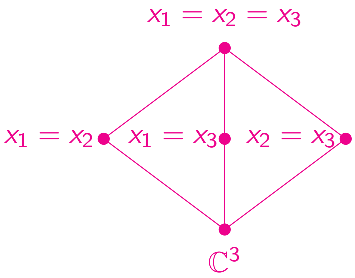

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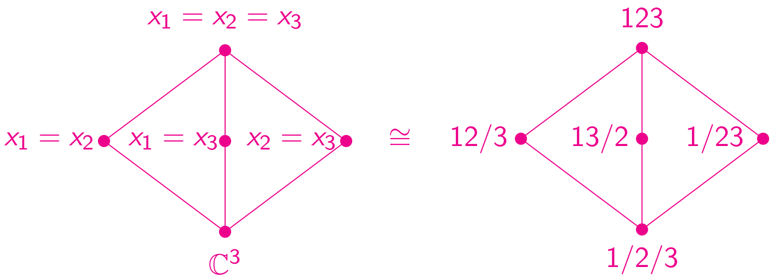

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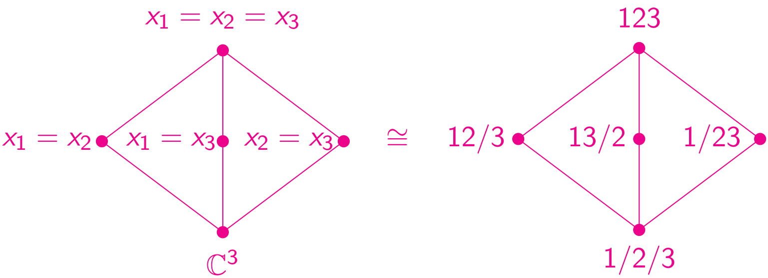

Proposition

We have $L(Br_n) \cong \Pi_n$ as posets.

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Theorem (SSS)

Let $G = G(m,p,n)$.

$$S(G,k) = \begin{cases} h_{n-k}(1, m+1, \ldots, km+1) := h(m,k,n) & \text{for } p < m, \\ h(m,k,n) - nh_{n-k-1}(m, 2m, \ldots, km) & \text{for } p = m. \end{cases}$$

THANKS FOR LISTENING!