A Family of Convex Sets in the Plane with the $(4, 3)$-Property can be Pierced by 9 Points

Daniel McGinnis

Iowa State University

Helly's Theorem

Helly's Theorem

Definition

A family of sets is said to be pierced by a set S if each set in the family contains a point in S.

Helly's Theorem

Definition

A family of sets is said to be pierced by a set S if each set in the family contains a point in S.

Theorem (Helly's Theorem (1913))

Let $\mathcal{F}$ be a finite family of compact, convex sets in $\mathbb{R}^d$ such that every $d+1$ sets in $\mathcal{F}$ have a common point. Then $\mathcal{F}$ can be pierced by 1 point.

(p, q)-property

Definition

A family $\mathcal{F}$ satisfies the (p, q)-property if for every p sets of $\mathcal{F}$, q of them have a common point.

(p, q)-property

Definition

A family $\mathcal{F}$ satisfies the (p, q)-property if for every p sets of $\mathcal{F}$, q of them have a common point.

Problem (Hadwiger and Debrunner (1957))

Let $p \geq q \geq d + 1$. Does there exist a constant $c_d(p, q)$ such that every finite family $\mathcal{F}$ of compact, convex sets in $\mathbb{R}^d$ satisfying the (p, q)-property can be pierced by $c_d(p, q)$ points?

(p, q)-property

Definition

A family $\mathcal{F}$ satisfies the (p, q)-property if for every p sets of $\mathcal{F}$, q of them have a common point.

Problem (Hadwiger and Debrunner (1957))

Let $p \geq q \geq d + 1$. Does there exist a constant $c_d(p, q)$ such that every finite family $\mathcal{F}$ of compact, convex sets in $\mathbb{R}^d$ satisfying the (p, q)-property can be pierced by $c_d(p, q)$ points?

Theorem (N. Alon, D. Kleitman (1992))

There is such a constant $c_d(p, q)$.

(p, q)-property

Definition

A family $\mathcal{F}$ satisfies the (p, q)-property if for every p sets of $\mathcal{F}$, q of them have a common point.

Problem (Hadwiger and Debrunner (1957))

Let $p \geq q \geq d + 1$. Does there exist a constant $c_d(p, q)$ such that every finite family $\mathcal{F}$ of compact, convex sets in $\mathbb{R}^d$ satisfying the (p, q)-property can be pierced by $c_d(p, q)$ points?

Theorem (N. Alon, D. Kleitman (1992))

There is such a constant $c_d(p, q)$.

The smallest such constant is denoted by $HD_d(p, q)$.

Upper Bounds on $HD_2(4, 3)$

Upper Bounds on $HD_2(4,3)$

- The first non-trivial case is to determine $HD_2(4,3)$.

Upper Bounds on $HD_2(4,3)$

- The first non-trivial case is to determine $HD_2(4,3)$.
- Alon - Kleitman proof: $HD_2(4,3) \leq 243$.

Upper Bounds on $HD_2(4,3)$

- The first non-trivial case is to determine $HD_2(4,3)$.
- Alon - Kleitman proof: $HD_2(4,3) \leq 243$.
- However, ther best known lower bound is $HD_2(4,3) \geq 4$.

Upper Bounds on $HD_2(4,3)$

- The first non-trivial case is to determine $HD_2(4,3)$.
- Alon - Kleitman proof: $HD_2(4,3) \leq 243$.
- However, ther best known lower bound is $HD_2(4,3) \geq 4$.

Theorem (Gyárfás, Kleitman, Tóth (2001))

$HD_2(4,3) \leq 13$.

Main Theorem

Theorem (M. 2020+)

$HD_2(4, 3) \leq 9.$

KKM Theorem

- Let $\Delta^n = \text{conv}\{e_i \ : \ i \in [n+1]\}$ (the standard n-dimensional simplex)

KKM Theorem

- Let $\Delta^n = \mathrm{conv}\{e_i \ : \ i \in [n+1]\}$ (the standard n-dimensional simplex)
- Every $J \subset [n+1]$ corresponds to the face $\sigma_J = \mathrm{conv}\{e_i \ : \ i \in J\}$ of Δ^n.

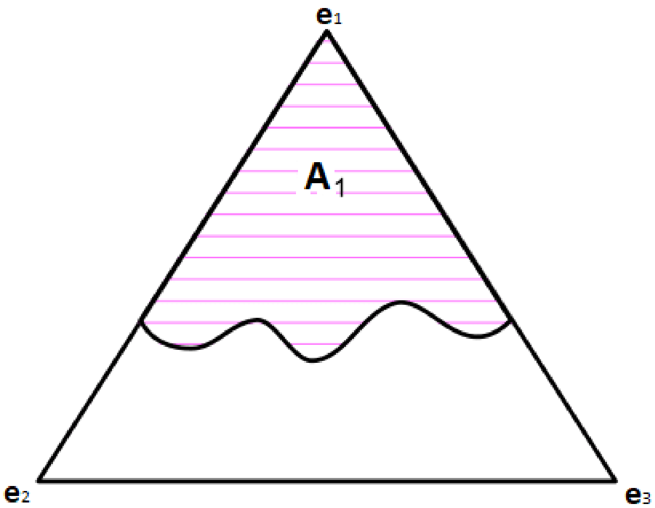
KKM Theorem

- Let $\Delta^n = \text{conv}\{e_i \ : \ i \in [n+1]\}$ (the standard n-dimensional simplex)
- Every $J \subset [n+1]$ corresponds to the face $\sigma_J = \text{conv}\{e_i \ : \ i \in J\}$ of Δ^n.

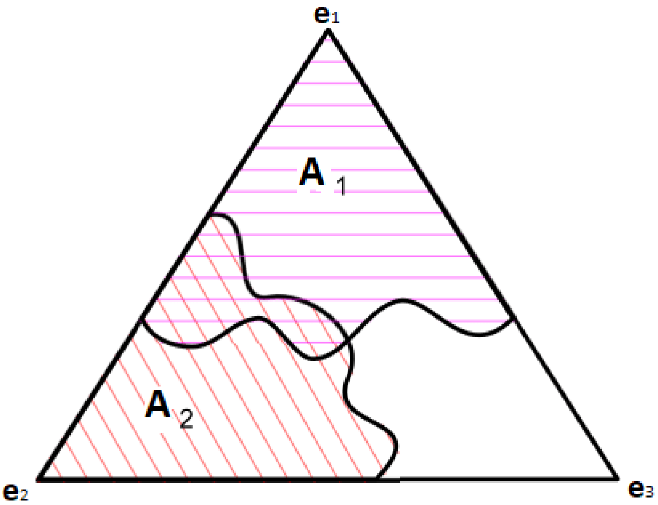
Theorem (KKM Theorem (1928))

Let $A_1, \dots, A_{n+1}$ be open subsets of Δ^n, such that $\sigma_J \subset \cup_{i \in J} A_i$ for all $J \subset [n+1]$. Then $\cap_{i=1}^{n+1} A_i \neq \emptyset$.

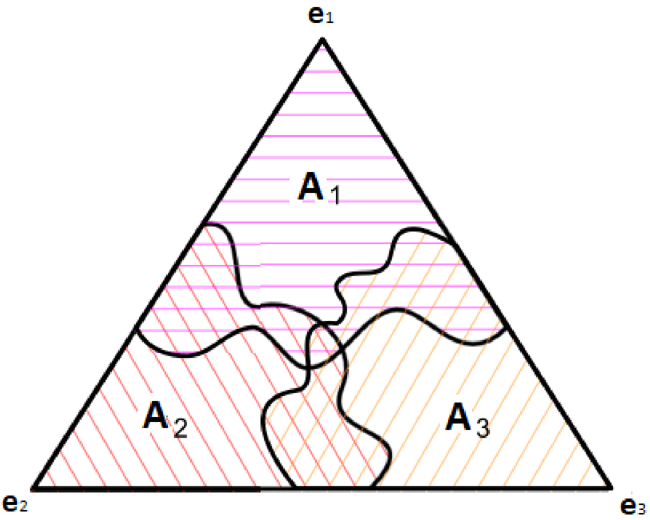
KKM Theorem



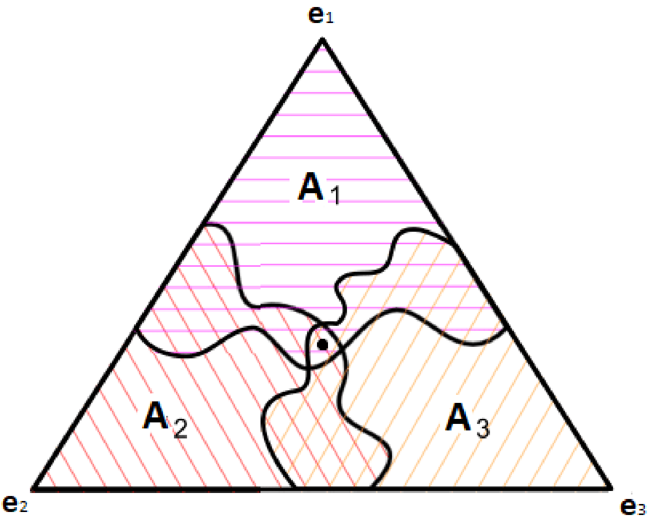
KKM Theorem



KKM Theorem

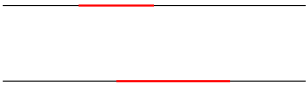


KKM Theorem



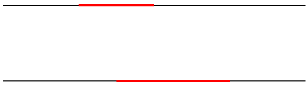
2-Interval Theorem

2-interval:



2-Interval Theorem

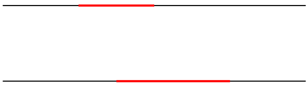
2-interval:



τ: piercing number

2-Interval Theorem

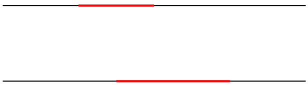
2-interval:



τ: piercing number
ν: matching number

2-Interval Theorem

2-interval:



τ: piercing number
ν: matching number

Theorem (Tardos (1995))

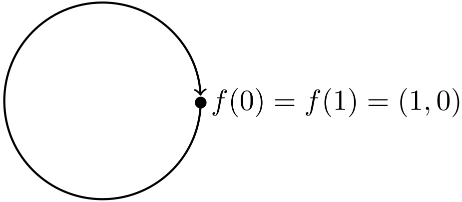
A family of 2-intervals satisfies $\tau \leq 2\nu$.

Using the KKM Theorem

- Let $\mathcal{F}$ be a finite family of compact, convex sets in $\mathbb{R}^2$ with the $(4,3)$-property.

Using the KKM Theorem

- Let $\mathcal{F}$ be a finite family of compact, convex sets in $\mathbb{R}^2$ with the $(4,3)$-property.
- Let $f : [0,1] \longrightarrow S^1$ be a parameterization of the unit circle

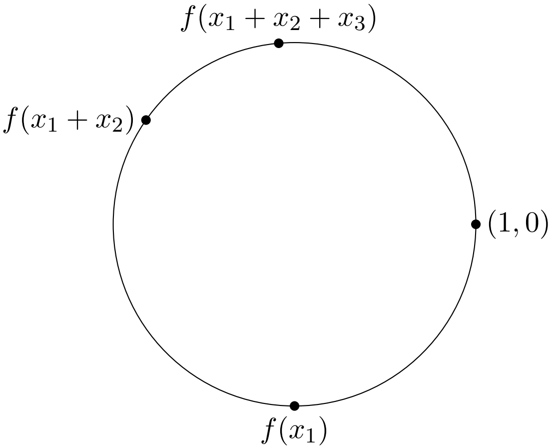


Using the KKM Theorem

Let $x = (x_1, x_2, x_3, x_4) \in \Delta^3$ (recall $x_1 + x_2 + x_3 + x_4 = 1$).

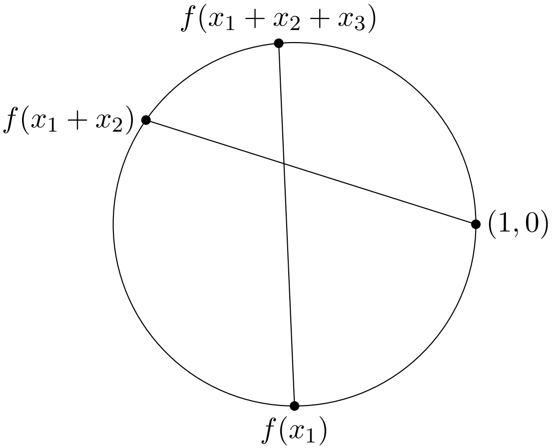
Using the KKM Theorem

Let $x = (x_1, x_2, x_3, x_4) \in \Delta^3$ (recall $x_1 + x_2 + x_3 + x_4 = 1$).



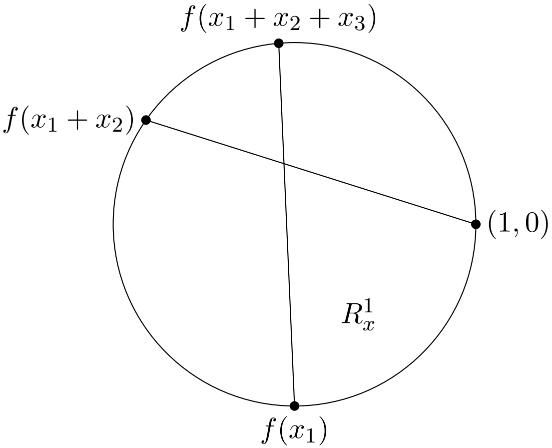
Using the KKM Theorem

Let $x = (x_1, x_2, x_3, x_4) \in \Delta^3$ (recall $x_1 + x_2 + x_3 + x_4 = 1$).



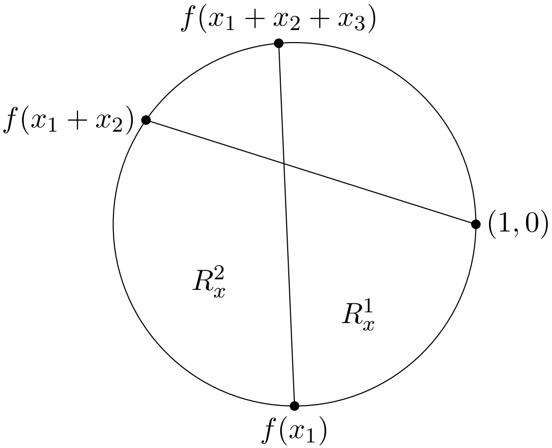
Using the KKM Theorem

Let $x = (x_1, x_2, x_3, x_4) \in \Delta^3$ (recall $x_1 + x_2 + x_3 + x_4 = 1$).



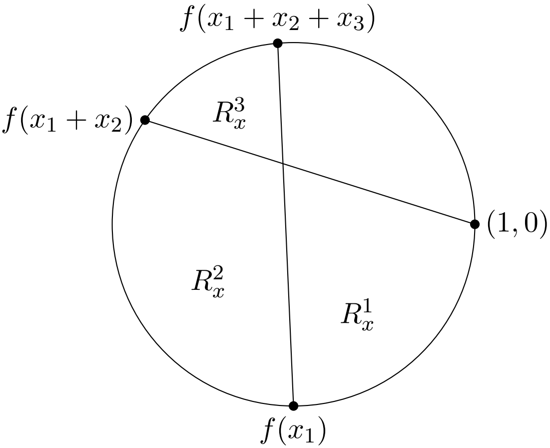
Using the KKM Theorem

Let $x = (x_1, x_2, x_3, x_4) \in \Delta^3$ (recall $x_1 + x_2 + x_3 + x_4 = 1$).



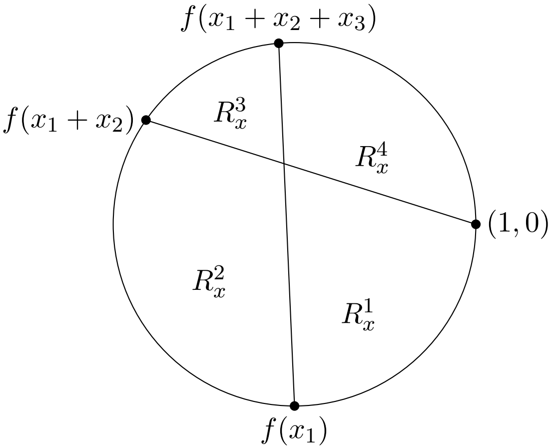
Using the KKM Theorem

Let $x = (x_1, x_2, x_3, x_4) \in \Delta^3$ (recall $x_1 + x_2 + x_3 + x_4 = 1$).



Using the KKM Theorem

Let $x = (x_1, x_2, x_3, x_4) \in \Delta^3$ (recall $x_1 + x_2 + x_3 + x_4 = 1$).

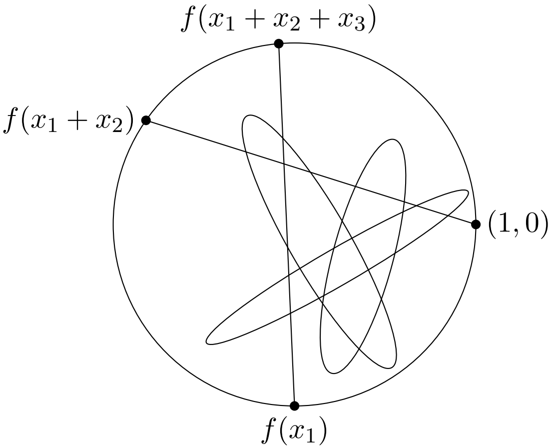


Using the KKM Theorem

We define the set A_i to contain a point $x \in \Delta^3$ whenever there are sets $F_1, F_2, F_3 \in \mathcal{F}$ such that $F_1 \cap F_2 \cap F_3 \neq \emptyset$ and $F_k \cap F_j \subset R_x^i$ when $k \neq j$.

Using the KKM Theorem

We define the set A_i to contain a point $x \in \Delta^3$ whenever there are sets $F_1, F_2, F_3 \in \mathfrak{F}$ such that $F_1 \cap F_2 \cap F_3 \neq \emptyset$ and $F_k \cap F_j \subset R_x^i$ when $k \neq j$.



Using the KKM Theorem

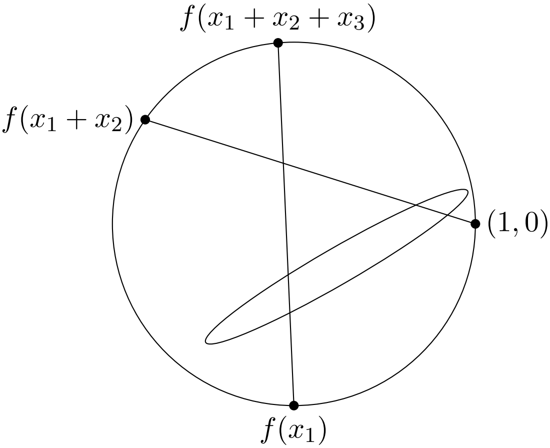
- First we consider the case when there is some $x \notin \cup_{i=1}^{4} A_i$

Using the KKM Theorem

- First we consider the case when there is some $x \notin \cup_{i=1}^{4} A_i$
- For $F \in \mathcal{F}$, we define a 2-interval on the two line segments

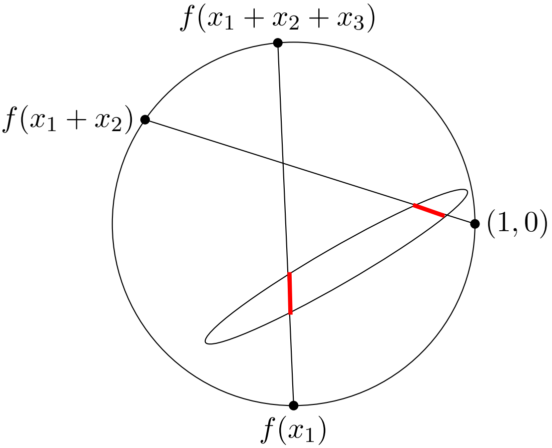
Using the KKM Theorem

- First we consider the case when there is some $x \notin \cup_{i=1}^{4} A_i$
- For $F \in \mathcal{F}$, we define a 2-interval on the two line segments



Using the KKM Theorem

- First we consider the case when there is some $x \notin \cup_{i=1}^{4} A_i$
- For $F \in \mathcal{F}$, we define a 2-interval on the two line segments



Using the KKM Theorem

- This family of 2-intervals has matching number at most 3.

Using the KKM Theorem

- This family of 2-intervals has matching number at most 3.
- $\mathcal{F}$ can be pierced by 6 points $(\tau \leq 2\nu)$

Using the KKM Theorem

- We can assume that $\Delta^3 \subset \cup_{i=1}^4 A_i$.

Using the KKM Theorem

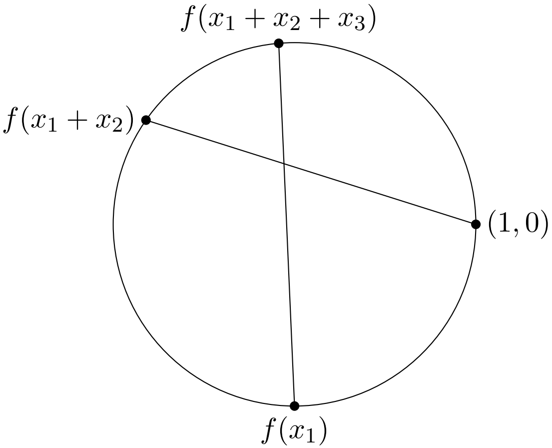
- We can assume that $\Delta^3 \subset \cup_{i=1}^{4} A_i$.
- In this case A_1, A_2, A_3, A_4 is a KKM cover so there exists $x \in \cap_i A_i$.

Using the KKM Theorem

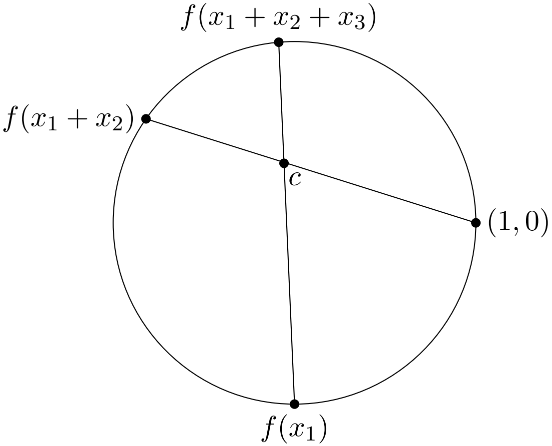
- We can assume that $\Delta^3 \subset \cup_{i=1}^4 A_i$.
- In this case A_1, A_2, A_3, A_4 is a KKM cover so there exists $x \in \cap_i A_i$.

From now on we fix a point $x \in \cap_i A_i$.

From now on we fix a point $x \in \cap_i A_i$.



From now on we fix a point $x \in \cap_i A_i$.



Piercing $\mathcal{F}_i$

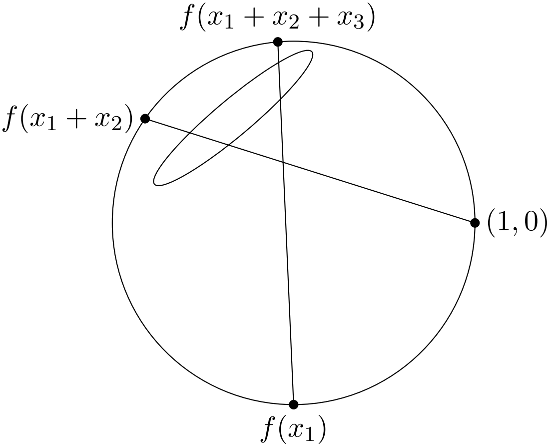
- Let $\mathcal{F}' \subset \mathcal{F}$ be the family of sets not pierced by c.

Piercing $\mathcal{F}_i$

- Let $\mathcal{F}' \subset \mathcal{F}$ be the family of sets not pierced by c.
- Let $\mathcal{F}_i \subset \mathcal{F}'$ be the sets that do not intersect R_x^i

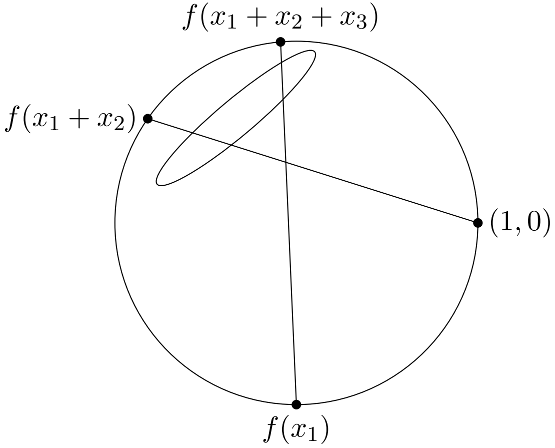
Piercing $\mathcal{F}_i$

- Let $\mathcal{F}' \subset \mathcal{F}$ be the family of sets not pierced by c.
- Let $\mathcal{F}_i \subset \mathcal{F}'$ be the sets that do not intersect R_x^i



Piercing $\mathcal{F}_i$

- Let $\mathcal{F}' \subset \mathcal{F}$ be the family of sets not pierced by c.
- Let $\mathcal{F}_i \subset \mathcal{F}'$ be the sets that do not intersect R_x^i



- We have that $\mathcal{F}' = \cup_i \mathcal{F}_i$

Piercing $\mathcal{F}_i$

- We show that $\mathcal{F}_1$ can be pierced by 2 points

Piercing $\mathcal{F}_i$

- We show that $\mathcal{F}_1$ can be pierced by 2 points
- Together with c, we can pierce $\mathcal{F}$ with 9 points.

Piercing $\mathcal{F}_i$

- We show that $\mathcal{F}_1$ can be pierced by 2 points
- Together with c, we can pierce $\mathcal{F}$ with 9 points.

Goal: find two lines so that the corresponding family of 2-intervals has matching number 1 $(\tau \leq 2\nu)$

Piercing $\mathcal{F}_i$

Let C_1, C_2, C_3 be three sets such that $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and their pairwise intersections are contained in R_x^1.

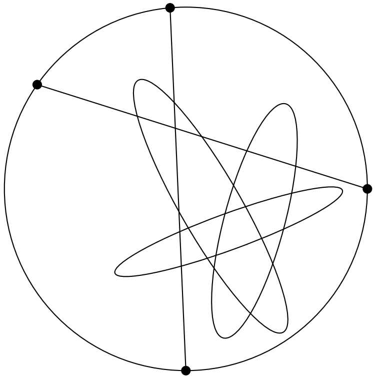
Piercing $\mathcal{F}_i$

Let C_1, C_2, C_3 be three sets such that $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and their pairwise intersections are contained in R_x^1.

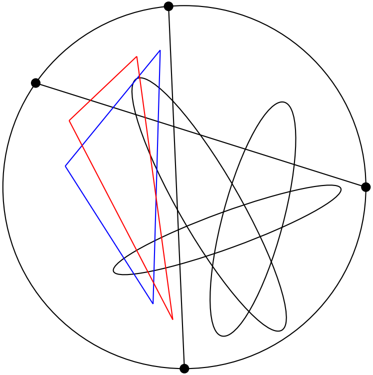
Lemma

Let $H, G \in \mathcal{F}_1$. Then $H \cap G$ intersects two of C_1, C_2, C_3.

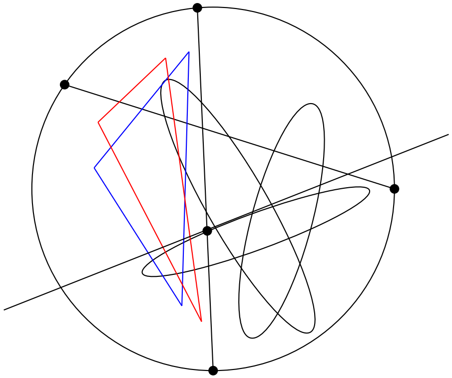
Piercing $\mathcal{F}_i$



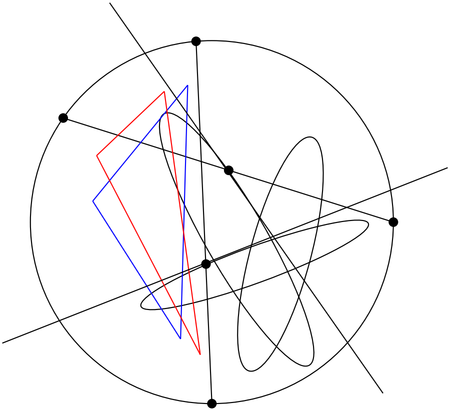
Piercing $\mathcal{F}_i$



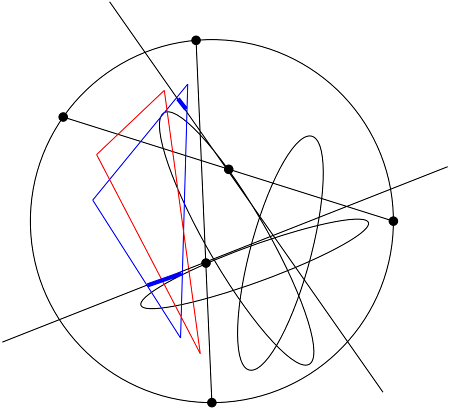
Piercing $\mathcal{F}_i$



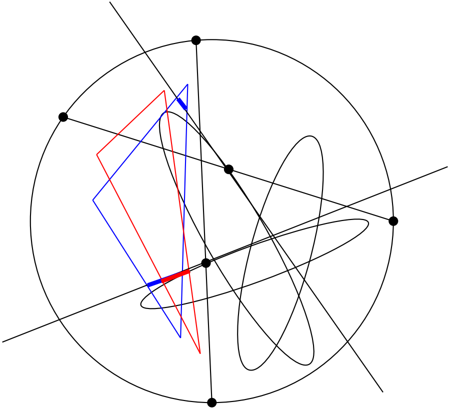
Piercing $\mathcal{F}_i$



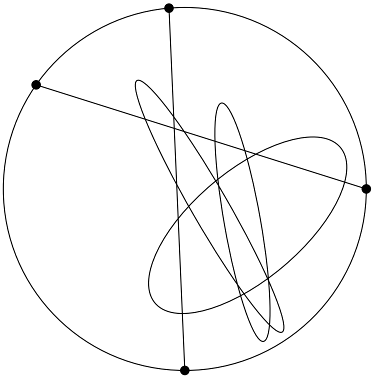
Piercing $\mathcal{F}_i$



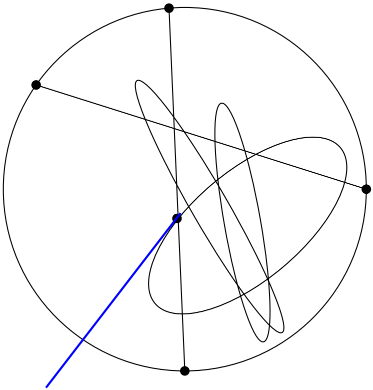
Piercing $\mathcal{F}_i$



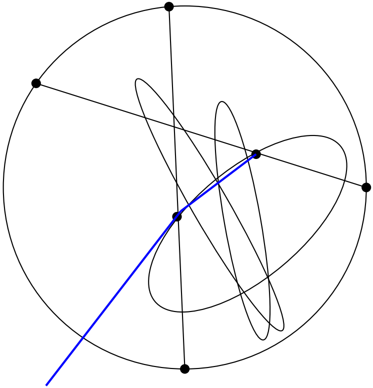
Piercing $\mathcal{F}_1$



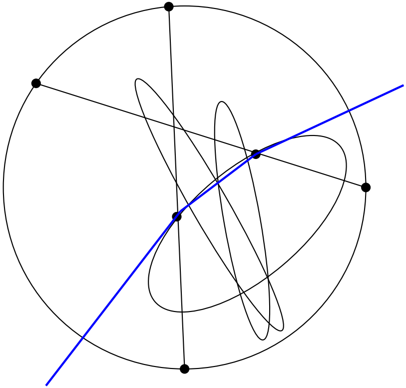
Piercing $\mathcal{F}_1$



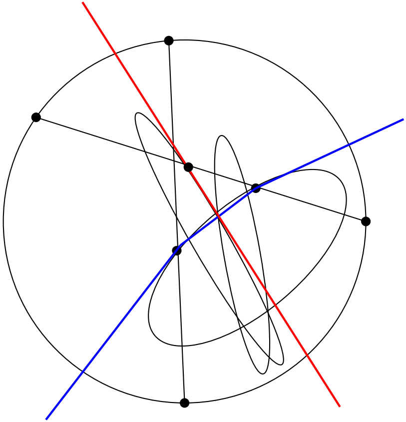
Piercing $\mathcal{F}_1$



Piercing $\mathcal{F}_1$



Piercing $\mathcal{F}_1$



Thank You!