A Family of Multijections

AMS Sectional Meeting

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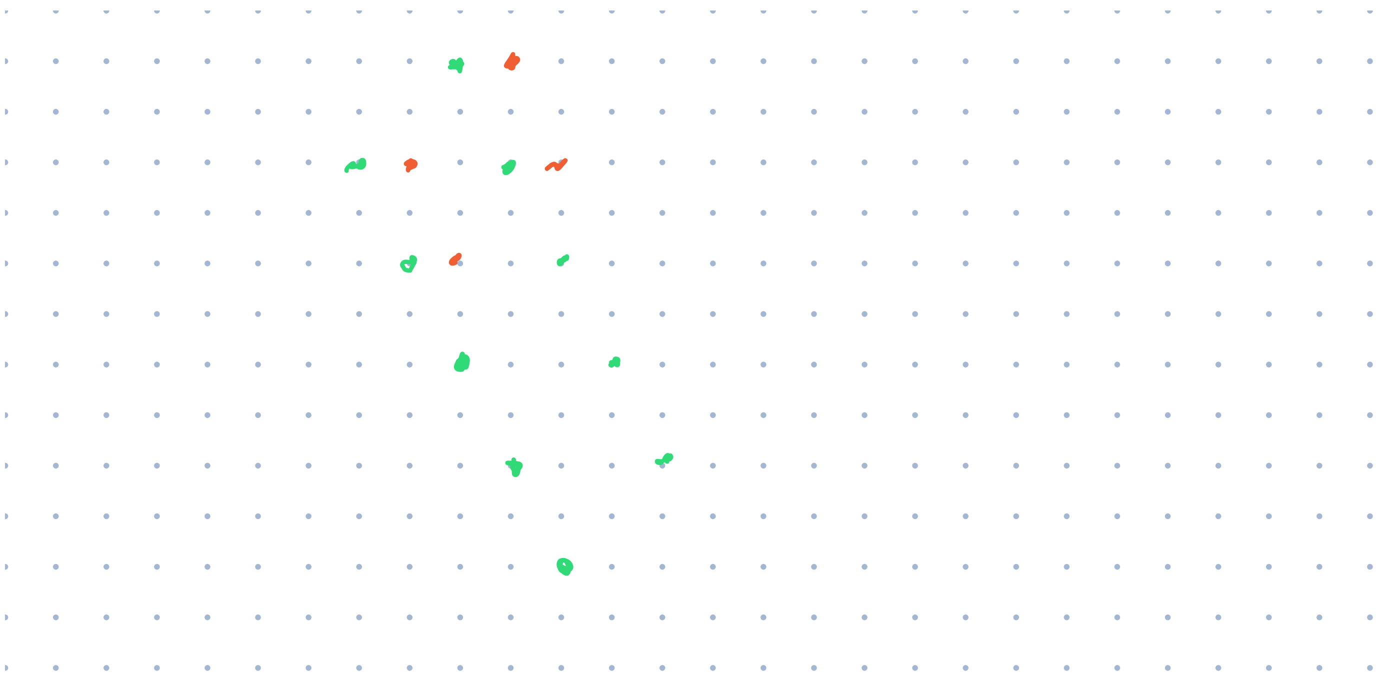
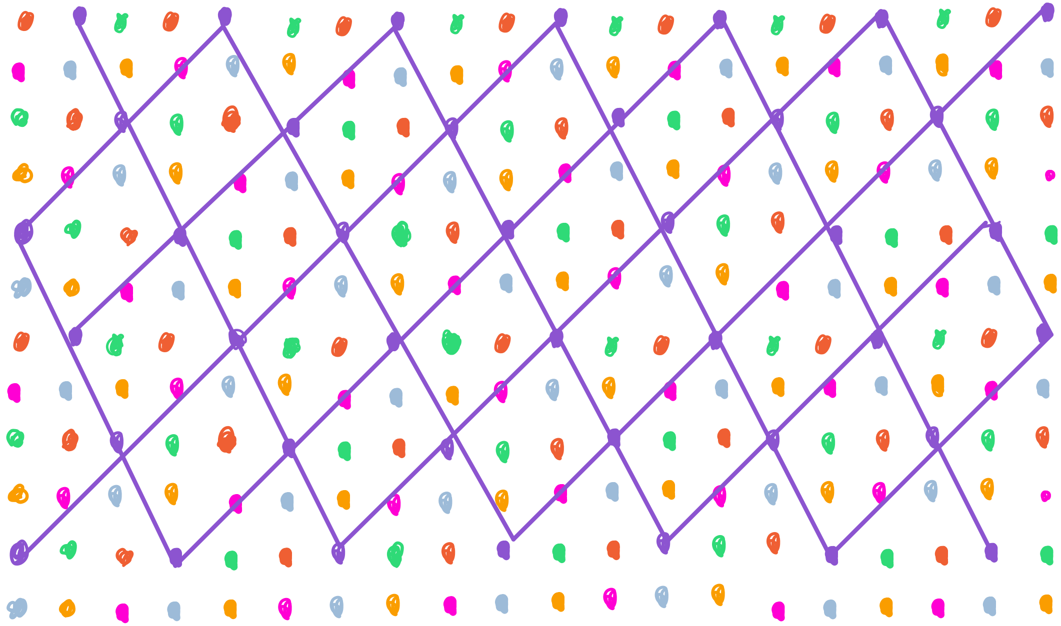
$$M = \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix} \qquad \mathrm{coker}_{\mathbb{Z}}(M) = \mathbb{Z}^2 / \mathrm{im}_{\mathbb{Z}}(M)$$

$$M\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ -2 \end{pmatrix} + b\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

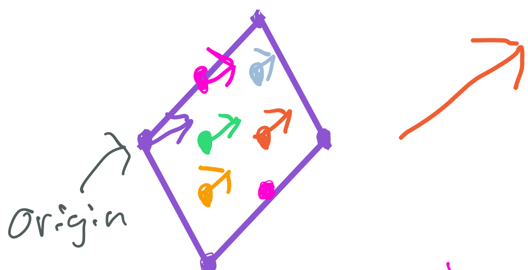


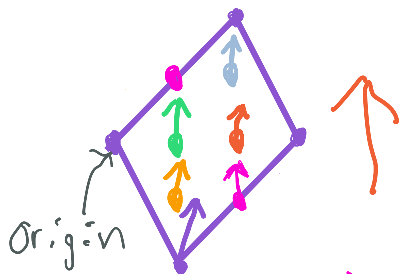
$$M = \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}$$



6 colors

1
2
3
4
5
6





reps for $\mathrm{Coker}_{\mathbb{Z}}(M)$ $=$

$(0,0)$	$(1,1)$
$(1,0)$	$(2,1)$
$(2,0)$	$(1,-1)$

or

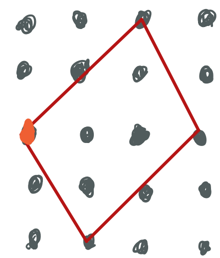
$(1,-2)$	$(2,-1)$
$(1,0)$	$(2,1)$
$(2,0)$	$(1,-1)$

etc.

Parallelepiped formed by Columns of M

$$\|$$

$$M = \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix} \longleftrightarrow \Pi_0(M) =$$



$$\mathrm{Vol}(\Pi_0(M))$$

$$\|$$

$$|M| = \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} = 1 \cdot 2 - 2 \cdot (-2)$$

$$2 + 4 = 6$$

$$\|$$

the size of $\mathrm{Coker}_{\mathbb{Z}}(M)$

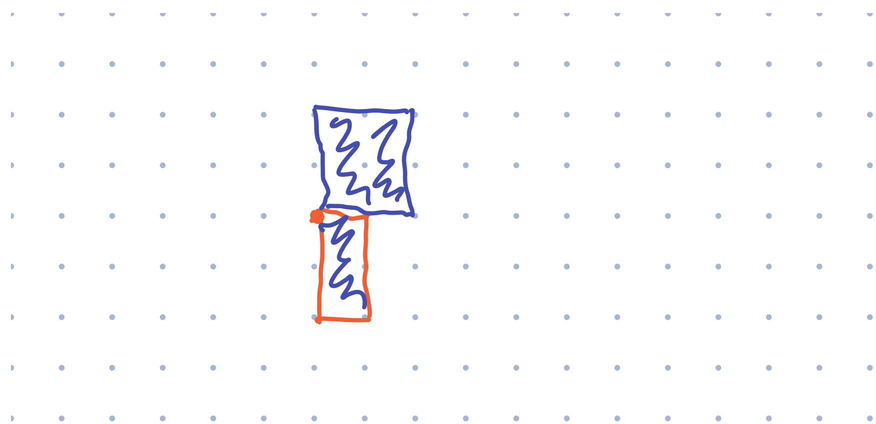
$$\begin{pmatrix} 1 & 2 \\ \hdashline -2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ \hdashline 2 & -2 \end{pmatrix}$$

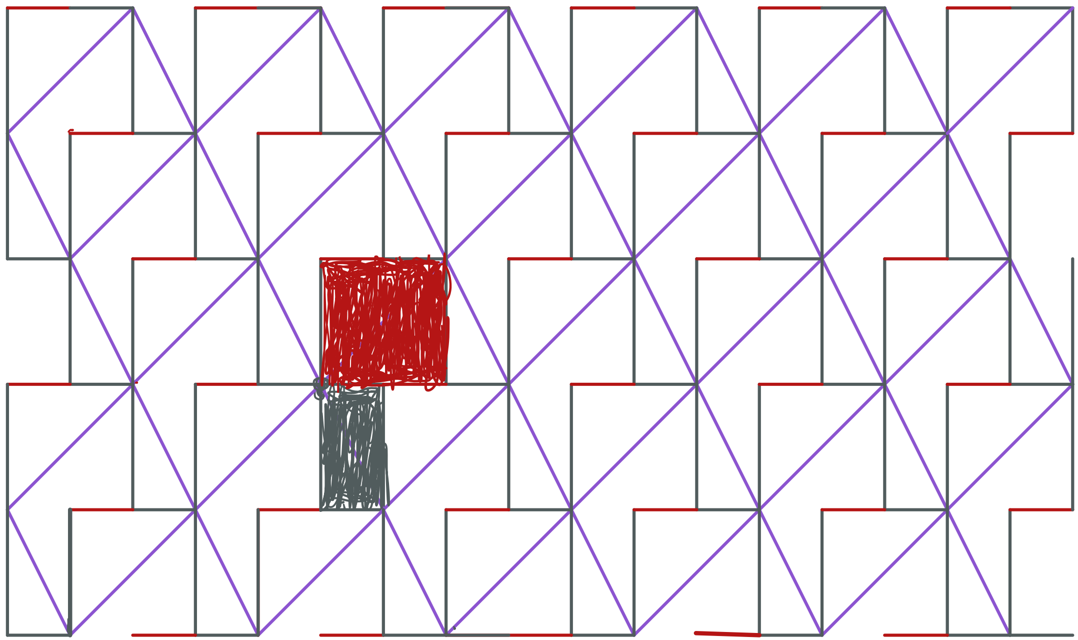
$T(M) :=$ "Tile" associated with M

$$M = \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}$$

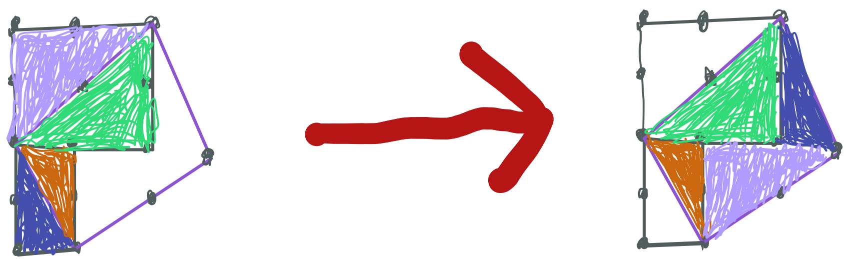
$$\Big\downarrow$$

$$T(M) = \Pi_{\bullet}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \cup \Pi_{\bullet}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$





$$M = \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}$$



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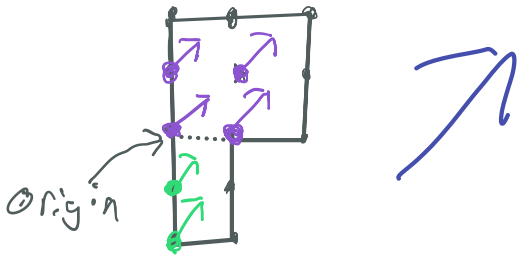
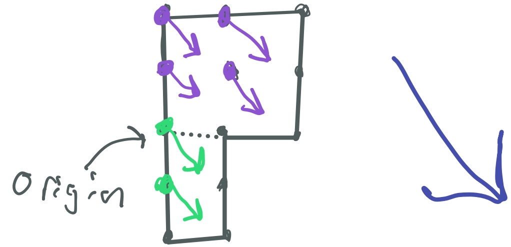




$$\text{reps for } \mathrm{Coker}_{\mathbb{Z}}(M) = \begin{array}{cc} (0,0) & (1,0) \\ (0,1) & (0,-1) \\ (1,1) & (0,-2) \end{array} \quad \text{or} \quad \begin{array}{cc} (0,1) & (1,2) \\ (0,2) & (0,0) \\ (1,1) & (0,-1) \end{array} \quad \text{etc.}$$

$$\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} = 2\cdot 1 + 2\cdot 2$$

$$2 + 4 = 6$$

Multiple row Laplace expansion

$$M = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$$\left|\operatorname{Coker}_{\mathbb{Z}} M\right| = |M| = \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

$$(-2) \cdot (-2) - (2) \cdot (-1) + (5) \cdot (1)$$

$$+ \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$$

$$+ (2) \cdot (4) - (4) \cdot (0) + (1)(2)$$

$$= 4 + 2 + 5 + 8 + 0 + 2 = 21$$

$$M = \left(\begin{array}{cccc} 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ \hline -2 & 0 & 1 & -1 \\ 0 & -1 & 0 & -2 \end{array}\right)$$

$$\downarrow$$

$$T(M) = \Pi_\bullet \left(\begin{array}{cccc} 3 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right) \cup \Pi_\bullet \left(\begin{array}{cccc} 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -2 \end{array}\right) \cup \Pi_\bullet \left(\begin{array}{cccc} 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \cup$$

$$\Pi_\bullet \left(\begin{array}{cccc} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right) \cup \Pi_\bullet \left(\begin{array}{cccc} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ \hline -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \cup \Pi_\bullet \left(\begin{array}{cccc} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ \hline -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

Multijection

$\mathrm{Coker}_{\mathbb{Z}}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \longrightarrow$ Summands in determinant expansion

Sandpile group of $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \longrightarrow$ Bases of matroid represented by K_1

Cutflow group of a certain cell complex $\longrightarrow$ Cellular spanning forests

Thm (M., 2021)

- If $M = \begin{pmatrix} K_1 \\ \hdashline K_2 \end{pmatrix}$ is an $(n \times n)$ square matrix such that K_1 and K_2 have complementary orientation*, then $T(M)$ periodically tiles $\mathbb{R}^n$.

- Furthermore, this tiling is obtained by translating $T(M)$ by the points in $\mathrm{im}_{\mathbb{Z}} M$.

* K_1 and K_2 have complementary orientation when the multi-row Laplace expansion for the determinant is a sum of all non-negative (or all non-positive) parts.

Thm (Chaiken, 1996)

K_1 is an $r \times n$ integer matrix
K_2 is an $(n-r) \times n$ integer matrix

TFAE:

1) K_1 and K_2 have complementary orientation.

2) The oriented matroid represented by K_1 is dual to the oriented matroid represented by K_2.

3) K_1 and K_2 have no common nonzero covectors.

$$K_1 = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

$a(3,2,1,1) + b(1,0,1,2)$

Cannot have the same sign pattern as

$$K_2 = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$c(2,0,-1,1) + d(0,1,0,2) \ \forall a,b,c,d \in \mathbb{R}^4 / (0,0,0,0)$

Matrices with Complementary Orientation

Suppose $K_1 = \begin{pmatrix} I_r & N \end{pmatrix}$ for some $r \times (n-r)$ matrix N.

Then, $K_2 = \begin{pmatrix} N^T & -I_{n-r} \end{pmatrix}$ has Complementary Orientation.

$$K_1 = \begin{pmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 1 & 2 & 6 & -4 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 3 & 2 & -1 & 0 & 0 \\ 5 & 6 & 0 & -1 & 0 \\ 7 & -4 & 0 & 0 & -1 \end{pmatrix}$$

Also, $M = \begin{pmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 1 & 2 & 6 & -4 \\ 3 & 2 & -1 & 0 & 0 \\ 5 & 6 & 0 & -1 & 0 \\ 7 & -4 & 0 & 0 & -1 \end{pmatrix}$ is Symmetric

$$K_1 = \begin{pmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 1 & 2 & 6 & -4 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 3 & 2 & -1 & 0 & 0 \\ 5 & 6 & 0 & -1 & 0 \\ 7 & -4 & 0 & 0 & -1 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 1 & 2 & 6 & -4 \\ 3 & 2 & -1 & 0 & 0 \\ 5 & 6 & 0 & -1 & 0 \\ 7 & -4 & 0 & 0 & -1 \end{pmatrix}$$

$$|M| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} \cdot \begin{vmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ -4 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ 0 & 6 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 & 0 \\ 6 & 0 & 0 \\ -4 & 0 & -1 \end{vmatrix}$$

$$1 \cdot (-1) - 2 \cdot 2 + 6 \cdot (-6)$$

$$- \begin{vmatrix} 1 & 7 \\ 0 & -4 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 & 0 \\ 6 & 0 & -1 \\ -4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 & 0 \\ 5 & -1 & 0 \\ 7 & 0 & -1 \end{vmatrix} - \begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} \cdot \begin{vmatrix} 3 & -1 & 0 \\ 5 & 0 & 0 \\ 7 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 7 \\ 1 & -4 \end{vmatrix} \cdot \begin{vmatrix} 3 & -1 & 0 \\ 5 & 0 & -1 \\ 7 & 0 & 0 \end{vmatrix}$$

$$-(-4) \cdot (-4) + (-3) \cdot (3) - (-5) \cdot (-5) + (-7) \cdot (7)$$

$$+ \begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} \cdot \begin{vmatrix} 3 & 2 & 0 \\ 5 & 6 & 0 \\ 7 & -4 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 7 \\ 2 & -4 \end{vmatrix} \cdot \begin{vmatrix} 3 & 2 & 0 \\ 5 & 6 & -1 \\ 7 & -4 & 0 \end{vmatrix} + \begin{vmatrix} 5 & 7 \\ 6 & -4 \end{vmatrix} \cdot \begin{vmatrix} 3 & 2 & -1 \\ 5 & 6 & 0 \\ 7 & -4 & 0 \end{vmatrix}$$

$$+ 8 \cdot (-8) - (-26) \cdot (-26) + (-62) \cdot 62$$

$$K_1 = \begin{pmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 1 & 2 & 6 & -4 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 3 & 2 & -1 & 0 & 0 \\ 5 & 6 & 0 & -1 & 0 \\ 7 & -4 & 0 & 0 & -1 \end{pmatrix}$$

K_1 and K_2 represent dual oriented matroids as well as dual arithmetic matroids.

Fact For every integer matrix, there are matrices representing the dual oriented arithmetic matroid.

In my thesis, I show how to construct all such matrices.

General Method

1) Consider K and $\hat{K}$ $r \times n$ integer matrices whose corresponding $r \times r$ minors match in sign (zeros are always okay)

$$K = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \end{pmatrix} \qquad \hat{K} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}$$

2) Choose a matrix $\hat{K}'$ that represents the oriented arithmetic matroid dual to $\hat{K}$

$$\hat{K}' = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}$$

3) K and $\hat{K}'$ have complementary orientations, so we can find a multijection!

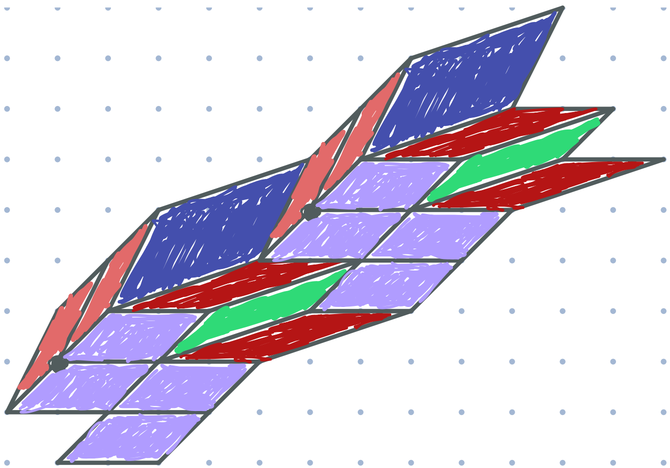
$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3 \cdot 1 - 4 \cdot (-1) + 5 \cdot 0$$

$$= 3 + 4 + 0 = 7$$

Lower-Dimensional Tiling

$$M = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ \hdashline 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

If M is $n \times n$, then $T(M)$ is in $\mathbb{R}^n$. However, there is a way to work in $\mathbb{R}^r$ instead (where r is the rows before the dotted blue line)



Rows and Columns

$\mathrm{Coker}_{\mathbb{Z}}(M)$ vs. $\mathrm{Coker}_{\mathbb{Z}}(M^T)$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \text{ vs } \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \text{ vs } \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Thm. Biggs, 1999

Motivation

The size of the Sandpile group of a graph is equal to the number of Spanning trees.

Sandpile
$\Updownarrow$ bijection
Spanning tree

Sandpile
$\Updownarrow$ multijection
Spanning forest

Thm. Duval-Klivans-Martin, 2015

For a d-dim'l cell complex Σ, the size of the Sandpile group of Σ is a weighted count of the cellular Spanning forests, where the weighting is given by the size of the torsion subgroup of the reduced relative (d-1)-dimensional homology group.

Further Directions

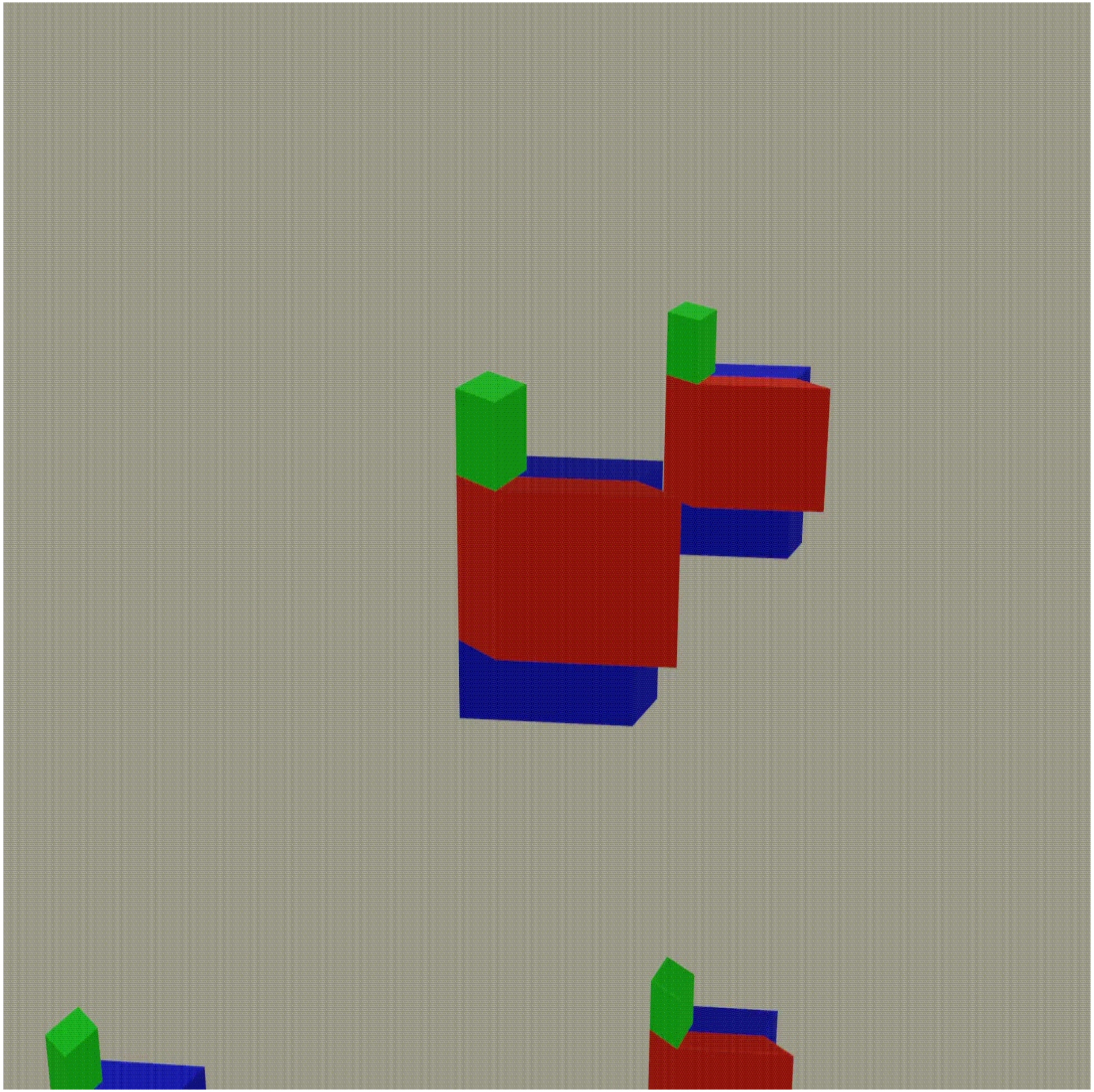
1) Does this multijection have applications to toric varieties or positroids?

2) Is there any way to generalize the multijection ideas to pairs of matrices without complementary orientation?

3) What can be said about the Pontryagin dual of the cell complex sandpile group?

4) Can Ehrhart theory say anything about $T(M)$?

5) Can we use the multijection to classify periodic tilings of $\mathbb{R}^n$ using rational parallelotopes?



Sources

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- Seth Chaiken Oriented matroid pairs, theory, and an electrical application Contemporary Mathematics 1996
- Art Duval, Caroline Klivans, and Jeremy Martin Cuts and flows of cell complexes Journal of Algebraic Combinatorics 2015
- Alex McDonough A family of matrix-tree multijections Algebraic Combinatorics 2021 (to appear)
- Alex McDonough Higher-dimensional sandpile groups and matrix-tree multijections PhD Thesis 2021