Iterated Matching Graphs

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Abstract

The matching graph $M(G)$ of a simple graph G has vertices which represent edges in G where two vertices in $M(G)$ are adjacent if and only if the corresponding edges in G do not share an endpoint. In this paper, we examine sequences of graphs generated by iterating the matching graph operation and characterize the behavior of various initial graphs. We show that a handful of graphs arrive at the empty set, two graphs "stabilize" and return themselves for each iteration, and all other graphs grow without bound while accumulating certain subgraphs.

1 Introduction

Over the course of this paper, we will be studying sequences of graphs generated by the matching graph operation. Given a simple graph G, its matching graph $M(G)$ has vertices that represent edges of G; two vertices in $M(G)$ are connected by an edge if and only if the corresponding edges of G are not incident. For readers familiar with graph theory, $M(G)$ is the complement of the line graph $L(G)$. In this paper, we completely characterize the end behavior of any graph under iteration of the matching graph operation.

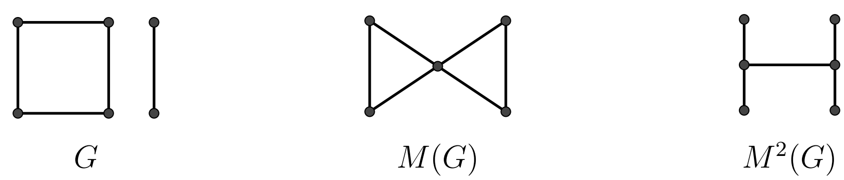


Figure 1: A graph G and its first and second matching graphs.

In Section 2, we define iterated matching graphs, dissipation, and snipped subgraphs. We then give some immediate results about these objects. Section 3 lists all graphs which dissipate, making a "tree" of graphs related by the matching graph operation (Figure 7). Graphs that do not dissipate are covered in Section 4. We present C_5 and the net graph as fundamental to our study (see Figure 8), introduce some useful tools for casework, and examine all connected graphs based on diameter. In Section 5, our work culminates in some truly fascinating results. In particular, we see that there is no graph except for C_5 or the net graph which gives itself for some iterated matching graph. We also show that if a graph does not dissipate and is not C_5 or the net graph, then the number of edges grows without bound as we continue taking the matching graph.

Towards the end of finalizing mathematical content, we became aware of a paper [CHJS97] which had a good deal of intersection with our work in its fourth section, though in very different language. The authors study the jump graph $J(G)$ which has an equivalent definition to our matching graph. In examining iterated jump graphs, they completely catalogue what we will later call d-finite graphs and recognize the importance of C_5 and the net graph. To prove these results, they define and use tools which are in some ways related to our notion of the snipped subgraph. This is where their analysis ended, though, whereas we will study more about the nature of d-infinite graphs. Specifically, our results stated in Theorem 5.1, Theorem 5.4, Theorem 5.5, and relating corollaries and lemmas do not appear in [CHJS97].

Our alternative name for the jump graph comes from the matching complex of a graph, an object studied and described as in [Jon08, Chapter 11] and [Wac03]. Our matching graph is the 1-skeleton of the matching complex. We were unable to find any other literature connecting the study of jump graphs and matching complexes, which both have a good deal of writing on them.

2 Preliminaries

A graph G is *simple* if any two vertices are connected by at most one edge and there are no edges from a single vertex to itself. An *isolated vertex* is a vertex that is not the endpoint of any edge. Throughout the paper, we assume that all graphs are simple, finite, and nonempty unless explicitly stated. Prior to performing the matching graph operation, we will often consider two graphs equivalent if they differ by only isolated vertices in light of Definition 2.1.

We denote the vertex set of a graph G by $V(G)$ and its edge set by $E(G)$, while we denote the number of vertices by $|V(G)|$ and the number of edges by $|E(G)|$. For all terms not defined here, see a standard graph theory textbook such as [Wes96].

Definition 2.1. Given a graph G, its *matching graph* $M(G)$ is the graph obtained by representing each edge of G by a vertex in $M(G)$. Two vertices in $M(G)$ are connected if and only if their corresponding edges in G are *not* incident.

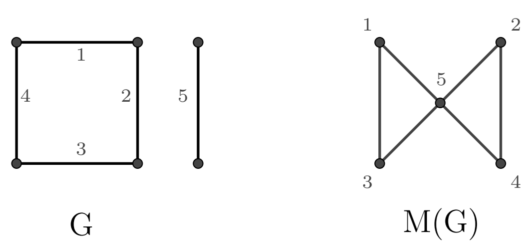


Figure 2: A graph G and its matching graph $M(G)$. Vertices in $M(G)$ are connected iff the corresponding edges in G do *not* share an endpoint.

Remark 2.2. This notion is closely related to that of the matching complex, which is more commonly studied; see, e.g., [Jon08, Chapter 11] and [Wac03]. Given a graph G, a *matching* is a set of edges such that no two edges in the set are incident. The *matching complex* of G is the set of all matchings in G. Note that the matching graph contains only the matchings of cardinality at most two. In this way, the matching graph is a particular subset of the matching complex, known as its 1-skeleton. Furthermore, a set of edges in G forms a matching if and only if the corresponding vertices in $M(G)$ form a clique, a subset of vertices such that all two have an edge between them. Thus the matching graph encodes all of the same information as the matching complex.

Remark 2.3. It should also be noted that there is another definition under the name "matching graph" as given in [ES98]. This notion is completely unrelated to our own.

Consider the connection between the matching graph and the well-studied *line graph* $L(G)$. Vertices of the line graph represent edges in G, and these vertices are connected with an edge if and only if the corresponding edges are incident. With this definition, we see that $M(G) = L(G)^c$, i.e., $M(G)$ is the the complement of $L(G)$. The line graph has been an important tool for approaching our study of matching graphs. We will use this observation in Section 4.

With the above definition of the matching graph, there are a few immediate results we can state about subgraphs. These are phrased in terms of induced subgraphs; a subgraph H of G is *induced* if there are no two vertices in H which are connected in G but not in H.

Lemma 2.4. *If H is a subgraph of G, then $M(H)$ is an induced subgraph of $M(G)$.*

Proof. Let H be a subgraph of G. Every edge in H is also an edge in G so $V(M(H)) \subseteq V(M(G))$. If there are two edges e_1, e_2 in H which are non-incident in H then they are also non-incident in G, and so $E(M(H)) \subseteq E(M(G))$. Together these imply $M(H) \subseteq M(G)$. Furthermore if there are two edges e_1, e_2 in H and G which are non-incident in G, then they must be non-incident in H. Thus $M(H)$ must be an induced subgraph of $M(G)$. □

Lemma 2.5. *If H and G are connected graphs, $M(G) \neq M(C_3)$, $M(H) \neq M(C_3)$, and $M(H)$ is an induced subgraph of $M(G)$ then H is a subgraph of G.*

Proof. Suppose $M(H)$ is an induced subgraph of $M(G)$. Then, since $M(G)^c = L(G)$, we know that $L(H)$ is an induced subgraph of $L(G)$. Because we assume that $M(H) \neq M(C_3)$, we know that $L(H) \neq L(C_3)$. Hence, by Whitney's Graph Isomorphism Theorem [Whi92], H is uniquely determined. By the same logic, since $M(G) \neq M(C_3)$ then G is uniquely determined as well. Then H must be a subgraph of G. □

Notice that Lemma 2.5 is a partial converse of Lemma 2.4. We can expand Lemma 2.5 to include some disconnected graphs with more conditions, but this is not relevant to the rest of the discussion. Lemma 2.4 will be much more useful moving forward.

Now, we turn attention towards the main object of study: *iterated matching graphs*.

Definition 2.6. Let G be a graph and define $M^0(G) = G$. For $k \geq 1$, the k^{th} *matching graph*, denoted $M^k(G)$, is the matching graph of $M^{k-1}(G)$.

Our goal is to study the behavior of the sequence $\{M^k(G)\}$. Of particular interest is determining whether a given graph dissipates in the following sense.

Definition 2.7. A graph G *dissipates* if there is some $k \geq 0$ such that $M^k(G) = \emptyset$. The *dissipation number*, denoted $d(G)$, is the smallest $k \geq 0$ such that $M^k(G) = \emptyset$. If there is no such k, then $d(G) = \infty$.

For examples of dissipation, see Figure 7. The reader can find the graph G from Figure 2 in the lower right-hand corner. By counting the arrows between G and the empty set, we see that $d(G) = 7$.

If $d(G) < \infty$, we will sometimes say that G is d-finite. Otherwise, we will say that G is d-infinite. In studying the behavior of $M^k(G)$, we will see that the following two notions play an integral part.

Definition 2.8. Suppose we have a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. A *quotient graph* Q of G is defined in the following way. Take some partition of $V(G)$ and then apply the equivalence relation created by this partition, letting $[v_i]$ denote such an equivalence class. The vertex set and edge set of Q are defined below.

$$V(Q) = \Big\{[v_i] : v_i \in V(G)\Big\} \qquad E(Q) = \Big\{\{[v_i], [v_k]\} : \{v_i, v_k\} \in E(G), [v_i] \neq [v_k]\Big\}$$

Note that, by definition, Q will not have any double edges or loops.

In a qualitative way, a quotient graph of a given graph is obtained by gluing vertices together and then deleting double edges and loops.

Figure 3: A graph G and a quotient Q of G. The colors represent our partition of $V(G)$.

Definition 2.9. A *snipped subgraph* of a graph G is a quotient graph of a subgraph of G.

Remark 2.10. In particular, any subgraph of G is also a snipped subgraph of G. This means G is a snipped subgraph of itself.

Above, we define a snipped subgraph by starting with G, taking a subgraph, and gluing together some of the vertices. This direction is helpful for understanding the definition, but the motivation is more clear if we conceptualize the snipped subgraph in another way. Suppose H is a snipped subgraph of G. Split apart the quotiented vertices of H and overlay it on top of G. Notice that the vertex-splitting action preserves disconnections among edges in H. Edges in $M(H)$ correspond to disconnections between edges in H. In this way, we know edges in $M(H)$ will be preserved under "snipping" of H. This property allows us to state the result in Lemma 2.11.

Figure 4 shows an example of a graph G and three of its snipped subgraphs. We can see how disconnections between edges in H_i are preserved under "snipping" of H_i. In graph H_1, edges 1 and 6 are non-incident and they are still non-incident in G. However, in graph H_2, edges 1 and 6 are incident, but these edges are non-incident in G.

The next lemma will show more explicitly our interest in snipped subgraphs. In the proof of Lemma 2.11, we will use the fact that if H is a quotient graph of G then there exists some subgraph $G' \subseteq G$ such that $|E(G')| = |E(H)|$ and H is a quotient graph of G'. We omit the proof of this, but it centers around considering the subset of edges of G which contribute to our construction of H and then simply removing any "unnecessary" edges in G which would give us double edges or loops if we considered the more qualitative notion of a quotient graph.

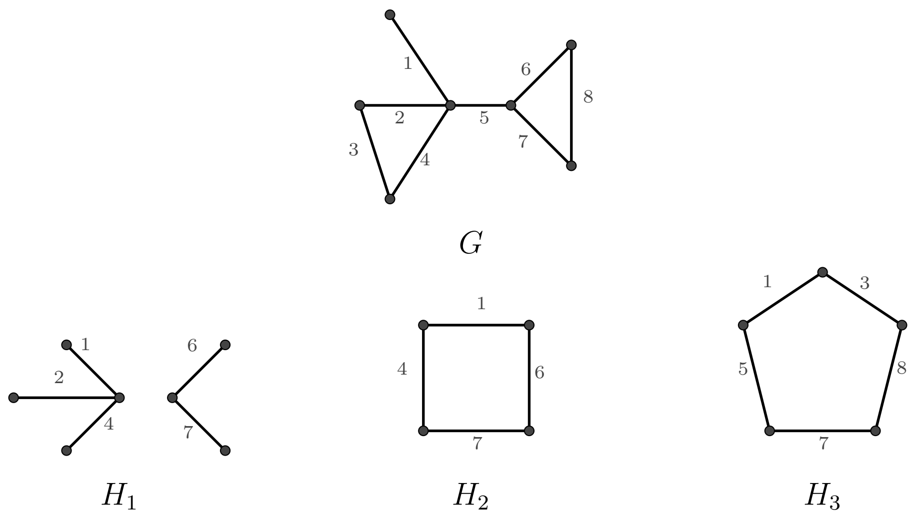


Figure 4: Graphs H_1, H_2, and H_3 are snipped subgraphs of G

Lemma 2.11. *If H is a snipped subgraph of G, then $M(H)$ is a subgraph of $M(G)$.*

Proof. Let G' be a subgraph of G such that H is a quotient graph of G' and $|E(G')| = |E(H)|$. For each edge $\{v_i, v_j\} \in E(G')$, we associate it with the unique edge $\{[v_i], [v_j]\} \in E(H)$.

There is at most one such edge in H identified with an edge in G'. If there were, say, two, we must have some edges $\{[v_1], [v_2]\}, \{[v_3], [v_4]\} \in E(H)$ both associated with $\{v_i, v_j\} \in E(G')$. Without loss of generality, this indicates that $[v_1] = [v_3] = [v_i]$ and $[v_2] = [v_4] = [v_j]$. By our definition of quotient graph, there are no double edges and so we must have $\{[v_1], [v_2]\} = \{[v_3], [v_4]\}$. Now because we also know $|E(G')| = |E(H)|$ we see that we must have each edge $e \in E(G')$ associated with *exactly* one, unique edge $e' \in H$ in this way.

For each vertex $e_{ij} = \{v_i, v_j\} \in M(G')$ call the associated vertex in $M(G)$ (using the above notions) $e'_{ij} = \{[v_i], [v_j]\}$. Now consider two distinct adjacent vertices $e'_{mn}, e'_{kl} \in M(H)$. Because H is a quotient graph of G' we know that e_{mn} and e_{kl} are adjacent in $M(G')$. We thus see that $M(H)$ is isomorphic to a subgraph of $M(G')$. We know that $M(G') \subseteq M(G)$ by Lemma 2.4 and so $M(H)$ is a subgraph of $M(G)$. □

Consider Lemmas 2.4 and 2.11 side by side. Lemma 2.4 gives us a way to recognize when we have a certain induced subgraph of $M(G)$ while Lemma 2.11 gives us a way to see when we have just a certain subgraph of $M(G)$. Because every *subgraph* is also itself a *snipped subgraph*, the set of subgraphs of G is contained in the set of snipped subgraphs of G. Hence, it is easier to satisfy the assumptions of Lemma 2.11 than those of Lemma 2.4. Because of this, and the fact that we do not need to consider induced subgraphs of $M(G)$ here, Lemma 2.11 will be very useful as we continue in our discussion.

Proposition 2.12. *If H is a snipped subgraph of G, then $d(H) \leq d(G)$.*

Proof. First, consider if $d(G) = \infty$. Then, trivially we have $d(H) \leq d(G) = \infty$.

Now, consider when $d(G)$ is finite and let $d(G) = d_0$. If $d_0 = 0$ then $G = \emptyset$ and the result is immediate. Thus, let d_0 be at least 1. Applying Lemma 2.11 iteratively, we conclude that

$$M^k(H) \subseteq M^k(G)$$

for all $k \geq 1$. Now, let $k = d_0$ and consider that

$$M^{d_0}(H) \subseteq M^{d_0}(G) = \emptyset.$$

And so we have $M^{d_0}(H) = \emptyset$ and it follows directly that

$$d(H) \leq d_0 = d(G).$$ □

Corollary 2.13. *If H is a snipped subgraph of G and $d(H) = \infty$, then $d(G) = \infty$.*

Corollary 2.13 reveals our primary use of snipped subgraphs. When presented with a graph G, we can look for any snipped subgraph H which we know to have $d(H) = \infty$. If we can find such a graph, we know that $d(G) = \infty$ as well. This makes it much simpler to determine if a given graph will dissipate or not. We will see Corollary 2.13 become very relevant in Section 4.

3 Graphs with finite d value

There are relatively few graphs with a finite d value. Notice that if G is d-finite then, for any $k \in \mathbb{N}$, $M^k(G)$ is also d-finite. Also, if $d(G) < \infty$, then by definition $M^d(G) = \emptyset$. In this way, we can build a tree of all d-finite graphs by "working backwards" from the empty set. First, we notice that $M^d(G) = \emptyset$ if $M^{d-1}(G)$ is either the empty set or a set of isolated vertices. We then take this latter possibility and consider what $M^{d-2}(G)$ must be. We continue doing this "backwards matching operation" until we are left with some $M^{d-n}(G)$ which cannot be the matching graph of any graph. In order to know when we have reached this point, we use Beineke's characterization of line graphs from [Bei70], in particular the information from Figures 5 and 6. In the tree of graphs (Figure 7), infinite families of graphs are blocked in grey and isolated vertices are not explicitly drawn where they may appear. For example, see that the graphs C_4 and C_4 with a diagonal edge lead to the same graph in Figure 7. We usually ignore isolated vertices because adding an isolated vertex to G does not change $M(G)$.

In his 1970 paper, Beineke showed that a graph is a line graph for another graph G if and only if it does not have one of the nine induced subgraphs in Figure 5 [Bei70]. Recall that $L(G)$ is the complement of $M(G)$. This implies that if some graph G is the matching graph of another graph G', then G^c must not have any of Beineke's nine forbidden induced subgraphs. Equivalently, G must not have the complement of any forbidden graph as an induced subgraph. These complements are shown in Figure 6.

While we only outline an idea for a proof here, Figure 7 does contains all d-finite graphs as shown rigorously in [CHJS97]. Hence, the reader can look at [CHJS97] if they would like to see the list in Figure 7 catalogued in rigor.

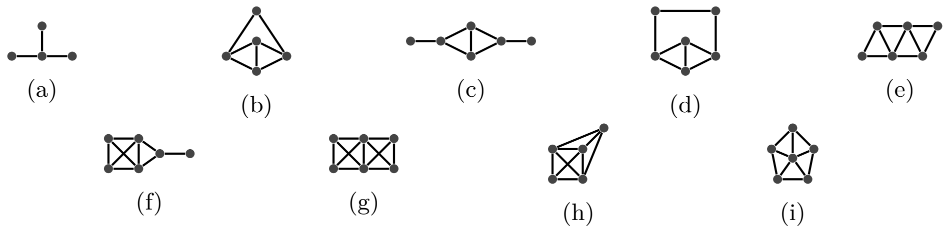


Figure 5: The nine graph which Beineke's theorem says cannot be an induced subgraph of any line graph.

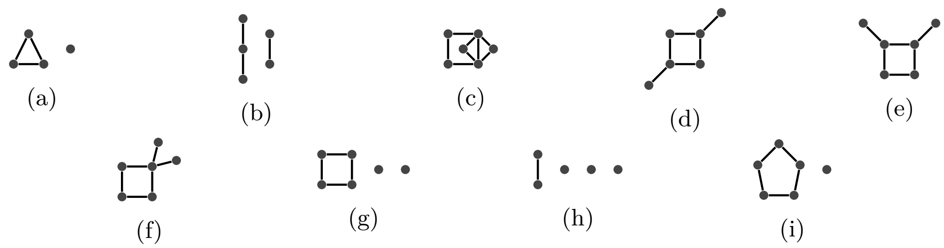


Figure 6: Complements of the nine graphs from Figure 5; equivalently, the graphs which cannot occur as induced subgraphs of any matching graph.

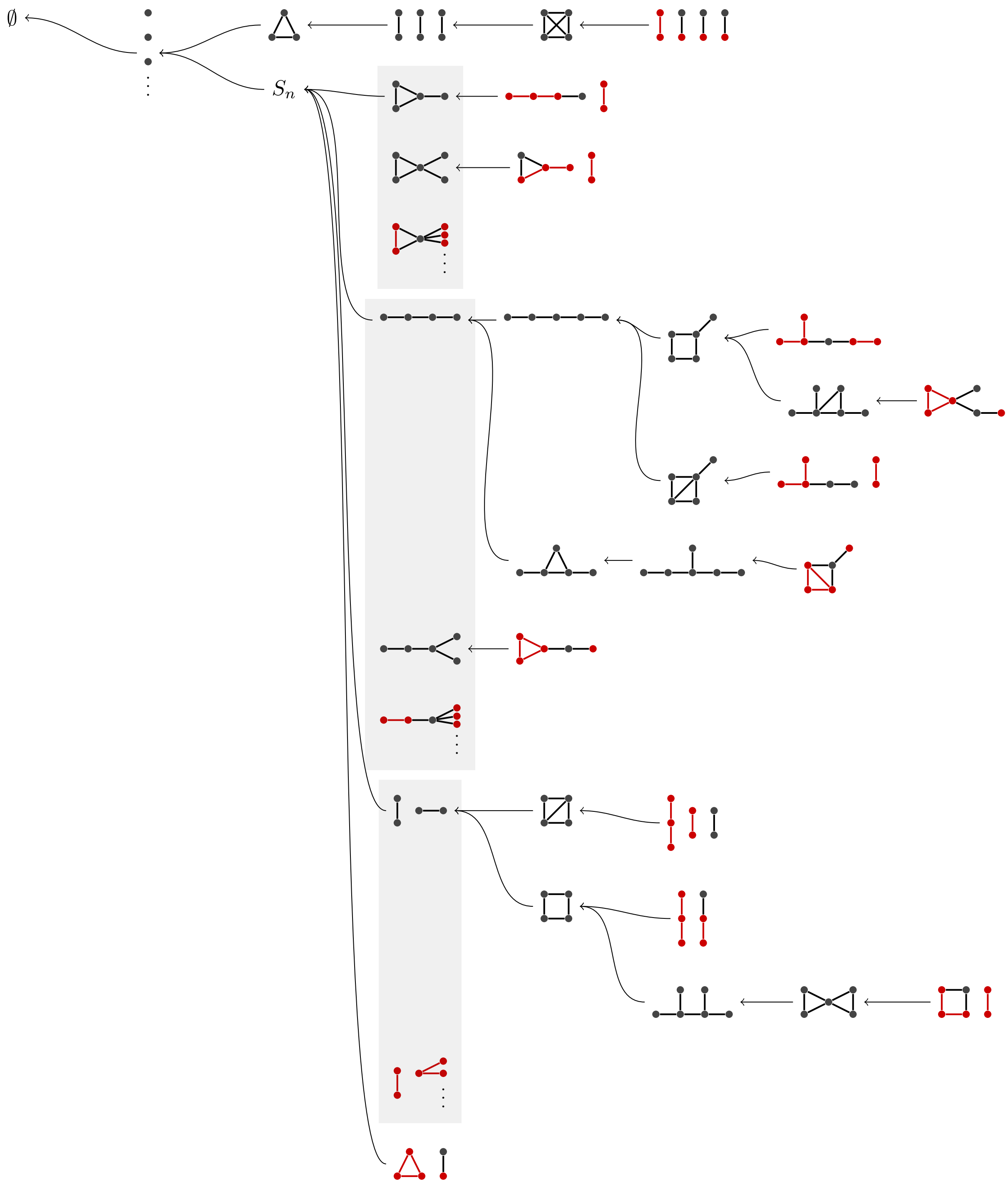


Figure 7: Graphs that dissipate. The arrows represent performing the matching graph operation. The d value of a given graph is found by counting the number of arrows from that graph to the empty set. Whenever a graph contains a forbidden subgraph from Figure 6, one such subgraph is highlighted in red.

4 Graphs with infinite d value

In this section, we turn to graphs with infinite d value. In some sense, Figure 7 already gives a characterization of d-infinite graphs—if G does not appear in Figure 7, then $d(G) = \infty$. However, there are more detailed questions we can answer about d-infinite graphs. In particular: what is the end behavior of these graphs? Is there a graph which satisfies $M^k(G) = G$ for some $k \geq 1$? In the following discussion, we will begin to answer these questions.

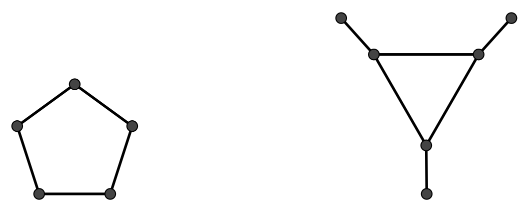
4.1 Two Special Graphs

As we have discussed earlier, the line graph $L(G)$ and the matching graph $M(G)$ are complements. This follows directly from the definition of each. Knowing this, we can apply the following result due to Aigner.

Theorem 4.1 ([Aig69]). *The only graphs satisfying $L(G)^c = G$ are C_5 and the net graph N.*

Applying this to iterated matching graphs, we have the following consequence.

Corollary 4.2. *If G is a graph, then $M^k(G) = G$ for all $k \geq 0$ if and only if G is C_5 or N.*



C_5 N

Figure 8: C_5 and the net graph N. These are the only graphs such that $G = M(G)$.

Since C_5 and N have themselves as matching graphs, if G has C_5 or N as a subgraph, then for every $k \geq 1$ we know C_5 or N is a subgraph of $M^k(G)$ by Lemma 2.4. Given this, we can talk about *accumulation*.

Definition 4.3. A graph G *accumulates* C_5 (resp. N) if there is some k such that $C_5 \subseteq M^k(G)$ (resp. $N \subseteq M^k(G)$).

Like with any d-infinite snipped subgraph, if G has C_5 or N as a snipped subgraph, then it will have infinite d value by Proposition 2.12. The surprising result is that the converse turns out to be true. If $d(G) = \infty$, then G will accumulate C_5 or N. In the following two subsections, we will focus on proving this fact which will be completed in Theorem 5.1.

4.2 Some Useful Graphs

The proofs in Section 4.3 involve a decent amount of casework. In order to make this more manageable, we will introduce some useful graphs in addition to C_5 and N. Each of the following graphs has an infinite d value and accumulates C_5. While performing casework in the next section, we can look for C_5, N, or one of these "useful graphs" as a snipped subgraph. If a graph G has one of these, then we know $d(G) = \infty$ and G accumulates C_5 or N.

Recall that Corollary 2.13 tells us that if H is a snipped subgraph of G and $d(H) = \infty$, then $d(G) = \infty$. Also, Lemma 2.11 says that if H is a snipped subgraph of G, then $M(H) \subseteq M(G)$. These two results will allow us to conclude that the following graphs have infinite d value and accumulate C_5.

First, we have $\mathbf{K_{2,3}}$ in Figure 9.

$K_{2,3}$ $M(K_{2,3})$

Figure 9: The graph $K_{2,3}$ and its matching graph.

The matching graph of $K_{2,3}$ has C_5 as a snipped subgraph. We conclude $d(K_{2,3}) = \infty$ and it accumulates C_5.

Next, we have the **Bug Graph** in Figure 10.

G M(G)

Figure 10: The bug graph and its matching graph.

The matching graph of the bug graph has $K_{2,3}$ as a snipped subgraph. We conclude the bug graph has $d(G) = \infty$ and it accumulates C_5.

Next, we have the **Stickman Graph** in Figure 11.

G M(G)

Figure 11: The stickman graph and its matching graph.

The matching graph of the stickman has $K_{2,3}$ as a snipped subgraph. We conclude the stickman has $d(G) = \infty$ and it accumulates C_5.

Lastly, we have the **Pendulum Graph** in Figure 12.

Figure 12: The pendulum graph and its matching graph.

The matching graph of this graph has $K_{2,3}$ as a snipped subgraph. We conclude the pendulum has $d(G) = \infty$ and it accumulates C_5.

Remark 4.4. All of the names for the above graphs (except for $K_{2,3}$) are original. There are more established names for the bug, stickman, and pendulum, but we decided against using them because of the memorable—and entertaining—nature of our own names.

4.3 Diameter and d-infinite graphs

In order to show that every graph with infinite d value accumulates C_5 or N, we will split all graphs into cases based on diameter. The *diameter* of a graph G is the length of the longest shortest path between any pair of vertices of G. The diameter of a disconnected graph is defined to be infinite.

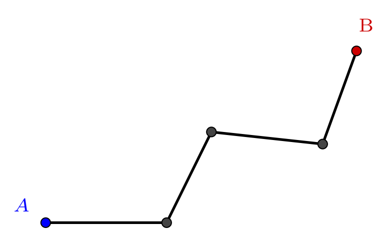
Lemma 4.5. *If a connected graph G has a diameter of 5 or more then G does not dissipate (i.e. $d(G) = \infty$) and G accumulates C_5.*

Proof. If G has a diameter of at least 5 then there is some path in G of length 5. Thus, we have C_5 as a snipped subgraph so $d(G) = \infty$ by Proposition 2.12 and $C_5 \subseteq M(G)$ by Lemma 2.11. □

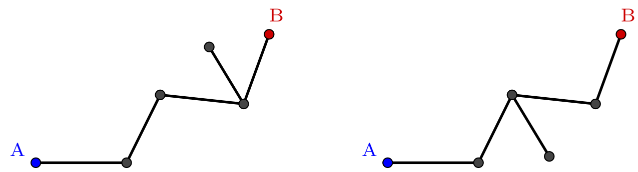
Lemma 4.6. *Suppose G has a diameter of* 4*. Then the following are equivalent:*

1. *G has an infinite d value,*
2. *G accumulates C_5 or N, and*
3. *G has* 6 *or more edges.*

Proof. Let P be a shortest path in G with a length of 4 and let P have endpoints A and B.



If G only has 4 edges, then $G = P$. In this case $d(G)$ is finite (see Figure 7). Now consider if G has 5 edges. Preserving the diameter constraint on G, the following graphs are the only two options for G.



We find that both these graphs have finite d value, as shown in Figure 7.

Now, let us consider when G has 6 or more edges. There must be two additional edges in G such that these edges and P form a connected subgraph of G. First, suppose that one of these edges is connected to either A or B. This edge cannot have another endpoint in P. If it did, then we would contradict the assumption that P is a shortest path from A to B. Therefore, this additional edge and P form a path of length 5 in G, giving C_5 as a snipped subgraph in G. This implies $d(G) = \infty$ and G accumulates C_5.

Thus, we turn our attention to the possibilities where neither additional edge is adjacent to A or B. There are two cases: only one of two edges is adjacent to a vertex in P, or both edges are.

In this first case, we immediately have C_5 as a snipped subgraph, since we have a path of length 4—given by P—disjoint from a single edge. One subgraph of this case is shown in Figure 13, with the snipped C_5 in gold. So in this case, G has an infinite d value and accumulates C_5.

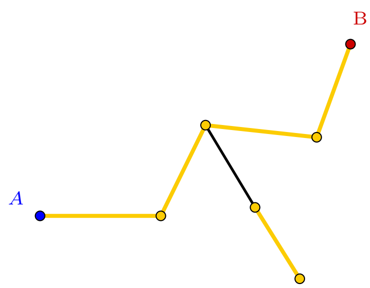
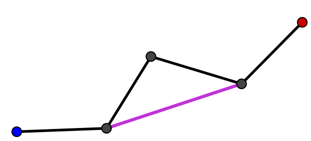


Figure 13: If G has two additional edges connected to P and only one is adjacent to a vertex in P, then G will have C_5 as a snipped subgraph.

Now, we address the second case: suppose that each additional edge has an endpoint which lies in P. The possibilities are restricted by the assumption that P is a shortest path between A and B. Adding the edges cannot create an alternative path from A to B that has a length less than 4. For instance, we cannot an edge to P in the following way.



There are two ways to add the additional edges so that they form a cycle in G, and there are four possibilities for adding two edges that do not form a cycle. The reader can look at any exhaustive list of small graphs such as that in [Ste] to confirm this.

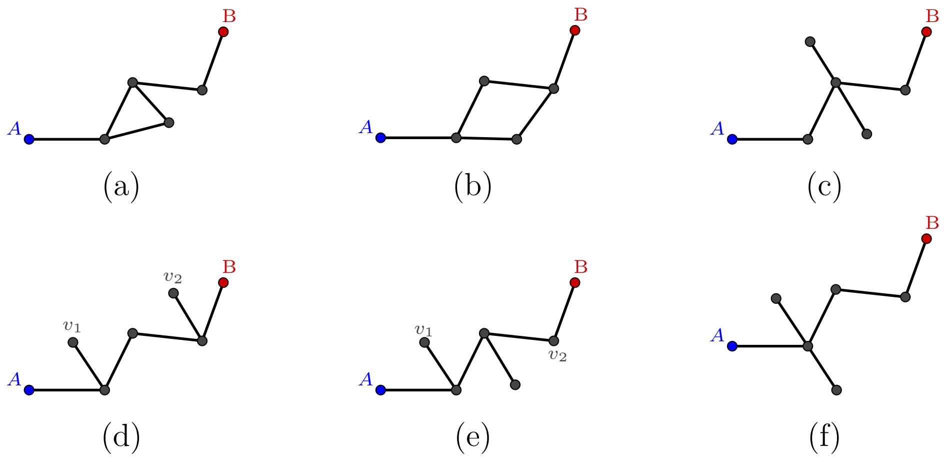


Figure 14: There are six ways to add two edges to P such that both are adjacent to a vertex in P which is not A or B and so that P remains a shortest path from A to B.

We claim that each of these graphs has an infinite d value and accumulates C_5 or N. First, graph (a) has P_5 as a subgraph and hence C_5 as a snipped subgraph. Next, both (b) and (d) have $K_{2,3}$ as a snipped subgraph. In (b) this can be seen by identifying vertices A and B. In (d), identify A and B with each other and v_1 and v_2 with each other. Graphs (c) and (f) both have the bug as a snipped subgraph. In both (c) and (f), identify vertices A and B to see this. Lastly, graph (e) has the net graph N as a snipped subgraph. To see this, identify the vertices labeled v_1 and v_2 with each other.

So in this second case, G will have infinite d value and accumulate C_5 or N, since we have shown this is true for graphs (a) - (f). Thus, if G has a diameter of 4 and at least 6 edges, it will have an infinite d value and accumulate C_5 or N. All graphs with diameter 4 and fewer than 6 edges will dissipate and therefore never accumulate C_5 or N. □

Lemma 4.7. *Suppose G has a diameter of 3. If $d(G) = \infty$, then it will accumulate C_5 or N. Furthermore, if G has diameter 3 and at least 7 edges, then $d(G) = \infty$ with the single exception of the family in Figure 15.*

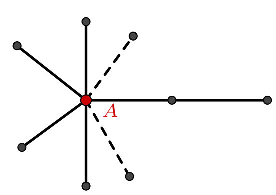
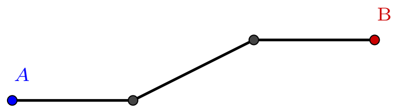


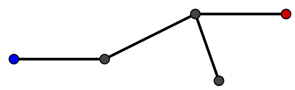
Figure 15: The only family of d-finite graphs with diameter 3 and at least 7 edges. Any number of additional pendants can be added off of vertex A.

Proof. Throughout this proof, whenever we say that a graph has finite d value "by explicit calculation," one can look back to Figure 7 to find the relevant calculation.

Let P be a shortest path in G with length 3 and endpoints A and B.



If G has 3 edges, then $G = P$. This graph has a finite d value. If G has 4 edges, then it must be the following graph.



This graph has a finite d value as well. If G has 5 edges, then it can be one of five possible graphs shown in Figure 16. One can see that these are all the graphs with diameter 3 and 5 edges by referencing a list of small graphs such as that given in [Ste].

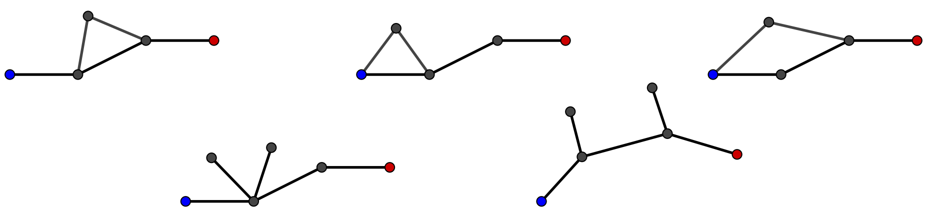


Figure 16: All graphs with diameter 3 and 5 edges.

Each graph in Figure 16 has finite d value. We can see this through explicit calculation.

Now, let us consider the graphs with diameter 3 and exactly 6 edges. The possibilities, up to isomorphism, are shown in Figure 17. Again, the reader can confirm that these are all options by referencing [Ste].

Graphs (a) through (d) in Figure 17 have a finite d value which can be seen by explicit calculation. We will show that graphs (e) through (k) each have infinite d value and accumulate C_5 or N: (e) is the stickman, (f) is the bug graph, (g) is the pendulum, (h) has P_5 as a subgraph, (i) is the net graph N, (j) has C_5 as a subgraph, and (k) has P_5 as a subgraph. Therefore, if G has 6 edges then it either has finite d value, or $d(G) = \infty$ and G accumulates C_5 or N.

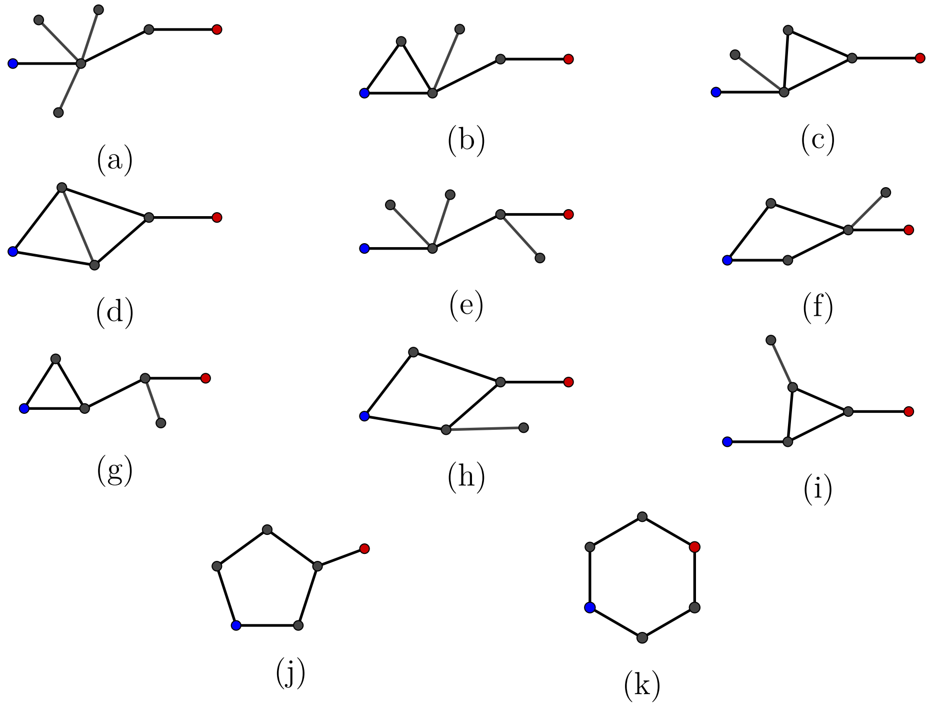


Figure 17: All graphs with a diameter of 3 and 6 edges. Graphs (a) - (d) dissipate while graphs (e) - (k) have infinite d-value.

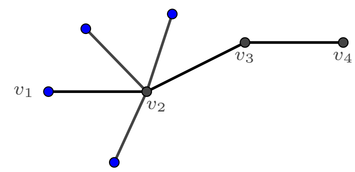
Now, suppose that G has diameter 3 and 7 or more edges. Then G must have a connected subgraph containing P and 3 additional edges. If this subgraph has a diameter of 5 or greater, then it will contain C_5 as a snipped subgraph. This means $d(G) = \infty$ and G accumulates C_5. If this subgraph has diameter 4, then by Lemma 4.6 we conclude that this subgraph—and consequently G—will have an infinite d value and accumulate C_5 or N. If this subgraph has diameter 3, then it must be one of the eleven graphs shown in Figure 17. The subgraph cannot have diameter less than 3 because we are assuming that P is a shortest path between A and B.

If the subgraph of G is isomorphic to graphs (e) through (k) in Figure 17, we have already shown that it is d-infinite and accumulates C_5 or N, so by Lemmas 2.13 and 2.4, $d(G) = \infty$ and G accumulates C_5 or N. Now, we will show that if G *strictly* contains any of the graphs (a) through (d), it will have $d = \infty$ and accumulate C_5 or N, except if G is part of the family shown in Figure 15.

Before beginning the casework, we will make two observations. Consider the picture of graph (a) below. Each of the blue vertices are identical under graph automorphism. Thus, without loss of generality, we will consider additions to graph (a) involving v_1 for all cases involving one blue vertex since all other blue vertices will follow the same logic as v_1. Now consider two graphs formed by adding an edge to graph (a). In the first, connect any non-adjacent vertices v_i and v_j with an edge. In the second, add a pendant edge to v_i. Notice that second graph has the first graph as a snipped subgraph. Because of this fact, we need not consider adding pendant edges to the graph unless there is a vertex which cannot be connected to any other vertex in the graph due to a diameter contradiction. These two observations will be applied to casework for graphs (b)-(d) as well.

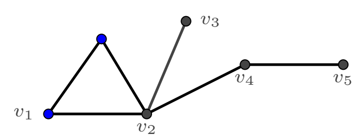
We will now examine graphs (a) through (d) from Figure 17 separately.

Graph (a)



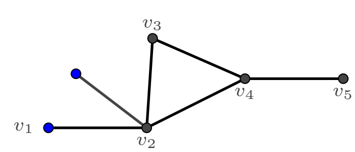
Now we can consider additions to graph (a). Adding an edge between v_1 and v_4 will give the bug as a subgraph, and adding an edge between v_1 and and v_3 gives the stickman as a subgraph. If we add an edge between two blue vertices, we again get the bug as a snipped subgraph. The only other two vertices in graph (a) that we can connect with an edge are v_2 and v_4. Adding this edge gives a graph with finite d value, but with diameter 2. This means that G must strictly contain this graph and fall into another case. Finally, we consider adding a pendant to v_2. This graph has finite d value and is part of the one exceptional family in Figure 15. So if G contains (a) as a strict subgraph and is not part of the family given in Figure 15, then $d(G) = \infty$ and G accumulates C_5.

Graph (b)



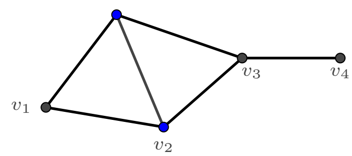
Like in the graph (a) case, we will consider v_1 in place of both blue vertices because the logic is interchangeable. We will again first consider ways to add an edge *between two vertices* in graph (b) before considering the addition of pendants. Adding an edge between v_1 and v_3 or v_1 and v_4 gives a path of length 5. Adding an edge between v_1 and v_5 gives C_5 as a subgraph. Adding an edge between v_2 and v_5 results in the bowtie graph with a pendant off the center vertex. The matching graph of this graph has $K_{2,3}$ as a snipped subgraph. If we add an edge from v_3 to v_4, we have a path of length 5. Finally, an edge between v_3 and v_5 results in a graph with the bug as a subgraph. Thus, if G contains graph (b) as a strict subgraph, it will have infinite d value and accumulate C_5.

Graph (c)



Next we consider ways that G can have graph (c) as a strict subgraph. Adding an edge between two blue vertices will give a path of length 5, and adding an edge between v_1 and v_3 also gives a path of length 5. If we connect v_1 and v_4 with an edge, we have $K_{2,3}$ as a snipped subgraph. Connecting v_1 and v_5 with an edge gives C_5 as a subgraph. Lastly, if we add an edge between v_2 and v_5 or v_3 and v_5 we have the bug as a subgraph. This tells us that if G contains graph (c) as a strict subgraph, it will be d-infinite and accumulate C_5.

Graph (d)



Lastly, we check the graph (d) case. If we add an edge between v_1 and v_3, the resulting graph is K_4 with a pendant edge off one vertex. The matching of this graph has $K_{2,3}$ as a snipped subgraph. Connecting v_1 and v_4 with an edge gives $K_{2,3}$ as a subgraph. Finally, if we add an edge between v_2 and v_4, we have C_5 as a subgraph. Hence, any G with graph (d) as a strict subgraph will have $d(G) = \infty$ and accumulate C_5.

This concludes the proof for the diameter 3 case. If G strictly contains any of graphs (a) through (d) and G is not part of the family given in Figure 15, then it will have a snipped subgraph with infinite d value which accumulates C_5. □

Lemma 4.8. *Suppose G has a diameter of 2. If $d(G) = \infty$, then G will accumulate C_5. Furthermore, if $d(G)$ is finite then G is one of the graphs in Figure 18.*

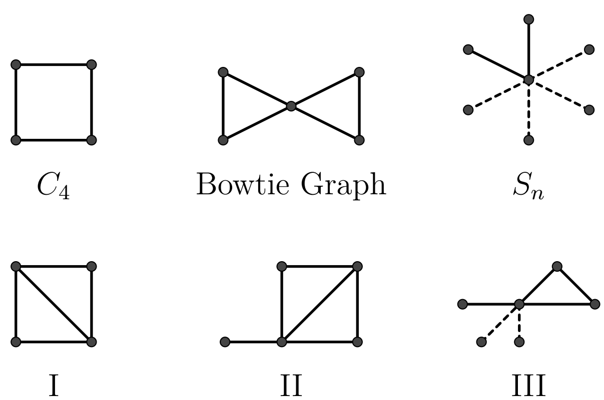
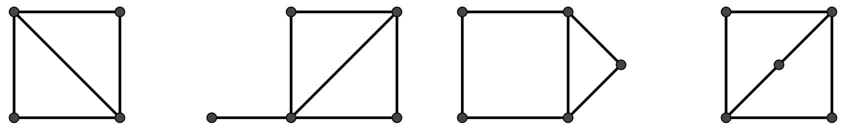


Figure 18: All graphs and families of graphs with a diameter of 2 which dissipate.

Proof. To start, we will prove the first part of the statement relating to d-infinite graphs. Suppose that G has a diameter of 2 and an infinite d value. First, assume that G has at least one cycle. If G has C_n where $n \geq 5$, then G will have C_5 as a snipped subgraph and by applying Lemma 2.11, we know that G will accumulate C_5. Now, suppose that G has C_4 as a subgraph. We know $d(C_4)$

is finite and so G must strictly contain C_4. It is straightforward to check that G must have one of the graphs in Figure 19 as a subgraph if it has C_4 as a strict subgraph and retains a diameter of 2.

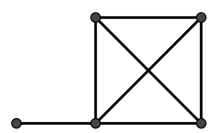


(i) (ii) (iii) (iv)

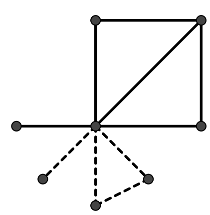
Figure 19: If $d(G) = \infty$, G has diameter 2, and $C_4 \subsetneq G$, then G has one of these as a subgraph.

Immediately we see that graph (iii) has C_5 as a subgraph. Also, we notice that graph (iv) is $K_{2,3}$. Hence, if G has either of these as a subgraph, then it will accumulate C_5.

Both graph (i) and graph (ii) have a finite d value. Thus, if G has either as a subgraph, it must be a strict subgraph because $d(G) = \infty$. There are only a few ways to add edges to (i) and (ii) so that they maintain a diameter of 2. Firstly, we could add edges to the C_4 as we have done in graphs (iii) or (iv). Either of these actions will reduce the case to what we have already studied and so G will accumulate C_5. Secondly, we could add edges to the C_4 as we have done in both graph (i) and graph (ii). This is shown in the following graph.

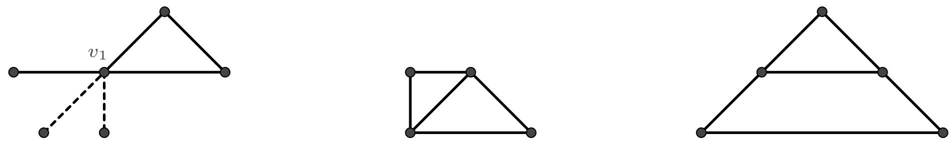


The matching graph of the above graph has $K_{2,3}$ as a snipped subgraph. Thus, if G has this above graph as a subgraph, then it will accumulate C_5. There are no other ways to add edges to graph (i) and preserve a diameter of 2 without reducing it to one of the other cases. So we need only to consider adding edges in other ways to graph (ii). There are two actions which we have not considered: adding more pendant edges to the vertex with one pendant or adding an edge between the endpoints of two pendants.



However, once we have two or more pendants, G will have the bug as a subgraph. So in this case, G will accumulate C_5. We have now considered all the possibilities when G has C_4 as a subgraph and $d(G) = \infty$, and in each one it accumulates C_5.

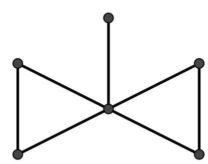
Now, suppose that G has C_3 as a subgraph. Again, C_3 has a finite d value, so G must contain it as a strict subgraph. It follows from straightforward casework that G must have one of the graphs in Figure 20 as a subgraph.



(v) (vi) (vii)

Figure 20: If $d(G) = \infty$, G has diameter 2, and $C_3 \subsetneq G$, then G has one of these graphs as a subgraph.

Notice that both graphs (vi) and (vii) have C_4 as a subgraph. We have already seen that if G has C_4 as a subgraph, diameter 2, and $d(G) = \infty$ then it will accumulate C_5. Hence, we only need to consider graph (v). This graph alone has a finite d value so if G contains it as a subgraph, graph (v) must be a *strict* subgraph. The only ways to add edges to graph (v) without forming a C_4 or arriving at a diameter contradiction are to add pendants to vertex v_1 or to add an edge between the endpoints of two pendants. If G is C_3 with only pendants off v_1, it will have a finite d value. If G is only the bowtie graph (see Figure 18), then it will also have a finite d value. However, if G contains graph (viii) below as a subgraph, then $d(G) = \infty$ and G will accumulate C_5, since the matching graph of (viii) contains the stickman as a subgraph.



(viii)

Thus, we know that if G has diameter 2, $d(G) = \infty$, and G has C_3 as subgraph, then it either has graph (viii) as a subgraph or it must also have C_4 as a subgraph. In all these cases, G will accumulate C_5.

Finally, we need to consider if G has no cycles. In this case, G must be a star graph. All star graphs have a finite d value, namely $d(S_n) = 2$. Therefore, we have shown the desired result. All graphs with a diameter of 2 and infinite d value must accumulate C_5.

We now show that the graphs and families of graphs listed in Figure 18 are the only d-finite graphs of diameter 2. Suppose G has diameter 2 and $d(G)$ is finite. Now suppose, to start, that G has at least one cycle. This cycle cannot be C_n for $n \geq 5$ because G is d-finite. Consider first if G has C_4 as a subgraph. We note that C_4 has finite d-value, as does graph (i) and graph (ii) from our above casework in this proof. From that same casework we see any other graph with diameter 2 and C_4 as a subgraph will have infinite d-value. This gives us C_4, graph I, and graph II in Figure 18.

Now suppose that G has C_3 as a subgraph but not C_4. If G is C_3 itself then it does not have diameter 2, which is a contradiction. Note that all graphs in the family given by graph (v) in Figure 20 have finite d-value. This gives family III in the list of d-finite graphs with diameter 2 (Figure 18). We then need to consider any ways we can add edges to C_3 without having C_4 as a subgraph or being in the family III while preserving finite d-value and diameter 2.

We consider how many C_3 subgraphs can appear in our graph. Nowhere can we have two C_3 subgraphs joined along an edge as we would then have C_4 as a subgraph. If we only have one C_3

then our graph must be in the family of graphs from III in Figure 18. See that we cannot have three (or more) C_3 graphs joined at a single vertex, because in this case our graph would have graph (viii) depicted above as a subgraph. We also cannot have any other combination of three or more C_3 graphs joined at different vertices. In such a graph, we have a path of length 5, a diameter contradiction, or C_4 as a subgraph, all of which are contradictions of our assumptions. We can, though, have two C_3 graphs joined at a vertex with no additional edges (this gives the bowtie graph in Figure 18). We cannot have an additional pendant edge off the shared central vertex because this would give graph (viii) above, which has infinite d value. We also cannot add an edge anywhere else since every other addition gives a path of length five. Thus, we see this case only adds the family of graphs in (v) (family III) and the bowtie graph.

Now suppose G has no cycle as a subgraph while still having diameter 2. This means that our graph must be a star graph and have finite d-value. This gives the last family, S_n, in Figure 18. □

5 Culminating Results

In Subsection 4.3, we showed that every connected d-infinite graph of diameter 2 or more will accumulate C_5 or N. In the following theorem, we will bring Lemmas 4.5, 4.6, 4.7, and 4.8 together to show that every d-infinite graph accumulates C_5 or N.

Theorem 5.1 (Accumulation Theorem)**.** *For any graph G, $d(G) = \infty$ if and only if G accumulates C_5 or N.*

Proof. First, suppose that G accumulates C_5 or N. Then there is some $k \geq 1$ such that $M^k(G)$ contains C_5 or N as a subgraph. It follows directly from Corollary 2.13 that $d(M^k(G)) = \infty$ and consequently, $d(G) = \infty$.

Next, assume that G is d-infinite. We will show that it must accumulate C_5 or N. Suppose first that G is connected. We have already shown that the statement is true for a diameter of 2 or more in Lemmas 4.5, 4.6, 4.7, and 4.8, so all that remains is when G has diameter 1. Notice that if $\operatorname{diam}(G) = 1$, then $G = K_n$ for some n. If $n < 5$ then $d(K_n)$ is finite as can be checked through explicit calculation (see Figure 7). For $n \geq 5$, we observe that K_n has C_5 as a subgraph. Thus, if G has diameter 1 and $d(G) = \infty$, it must have C_5 as a subgraph. Therefore, if G is a *connected* graph and $d(G) = \infty$, then G accumulates C_5 or N.

Now suppose G is *not* a connected graph. In particular say our connected components (of which there are at least two) are $\{H_i\}$, such that $H_i \cap H_j = \emptyset$ if $i \neq j$ and $\bigcup_i H_i = G$. We will assume G has no isolated vertices so each H_i contains at least one edge. We will now show that $M(G)$ is a connected graph. When we do this we will be able to use the argument from the first part of this proof on $M(G)$ and be done.

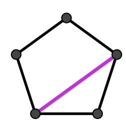
To see that $M(G)$ is connected, we consider arbitrary vertices $e_1, e_2 \in V(M(G))$ and will show that there is a path from e_1 to e_2. We have two cases to consider: either $e_1, e_2 \in H_k$ for some k, or $e_1 \in H_m$ and $e_2 \in H_n$ for $m \neq n$.

Consider our first case. Since we have at least two distinct connected components of G, say H_k and H_j, both with at least one edge, we have a path from e_1 to e_2 in $M(G)$ of length two. In particular we have the path $\{(e_1, e_j), (e_j, e_2)\}$ in $M(G)$, where $e_j \in E(H_j)$. Now suppose we have our second case. Then we see that e_1 and e_2 are actually adjacent in $M(G)$. We thus see that $M(G)$ is a connected graph. This means $M(G)$ accumulates C_5 or N and so G does as well. □

Now that we know every d-infinite graph accumulates C_5 or N, we can ask about the end behavior of the sequence $\{M^k(G)\}$ for a given graph G. We know that if G *is* C_5 or N, then the number of edges in $M^k(G)$ will stay constant as $k \to \infty$. But what happens for a d-infinite graph which is *not* C_5 or N? Is there such a G where the number of edges in $M^k(G)$ is constant? Or is there some G where $M^K(G) = G$ for $K > 1$, resulting in a cycle of iterated matching graphs? As it turns out, the answer to both these questions is "no". Every d-infinite graph which is not C_5 or N will grow without bound under the matching operation. We will spend the rest of this paper proving this fascinating result.

Lemma 5.2. *Suppose G has C_5 or the net graph N as a strict subgraph, then $|E(M^k(G))| \to \infty$ as $k \to \infty$.*

Proof. First, suppose that G has C_5 as a *strict* subgraph. Note that the number of edges in $M^k(G)$ cannot decrease. To see this, consider the following. Every edge in G must be non-incident to at least one edge in the C_5 because of the structure of C_5. For example, see that we cannot place the purple edge in Figure 21 so that it is incident with every edge. Hence, for each edge in G, we add at least one edge to $M(G)$. Therefore, $|E(G)| \leq |E(M(G))|$. If G has C_5 as a strict subgraph, $M(G)$ will as well and so we apply this logic iteratively to conclude $|E(M^k(G))| \leq |E(M^{k+1}(G))|$ for all $k \geq 1$. To show that the sequence is strictly increasing, we start by observing that if G has C_5 as a *strict* subgraph, then G must have the graph G_0 depicted in Figure 21 as a *snipped* subgraph.



G_0

Figure 21

Applying Lemma 2.4 iteratively, we can conclude that $M^k(G_0) \subseteq M^k(G)$ for all $k \geq 1$. Thus, if we show that $|E(M^k(G_0))| \to \infty$ as $k \to \infty$, then surely this will also be true for G.

The first and second matching graphs of G_0 are shown in Figure 22. We observe that the purple edge in $M(G_0)$ is non-incident to three other edges in $M(G_0)$. This accounts for the three purple edges in $M^2(G_0)$. Let H_0 be a subgraph of $M^2(G_0)$ containing the C_5 and only one purple edge; we see H_0 is exactly $M(G_0)$ and so $M^2(G_0)$ will be a subgraph of $M^3(G_0)$. This gives $|E(M^3(G_0))| \geq |E(M^2(G_0))|$. Now, let $E' \subseteq E(M^2(G_0))$ be the set of edges in $M^2(G_0)$ which are not in H_0. Because of our discussion at the beginning of this proof, every edge in E' must add at least 1 to the total edge count of $M^3(G_0)$. This gives

$$|E(M^3(G_0))| \geq |E(M^2(G_0))| + |E'| = |E(M^2(G_0))| + 2.$$

Hence, $|E(M^3(G_0))| - |E(M^2(G_0))| \geq 2$. Now, we can generalize this inequality for all $M^k(G)$. We know $M(G_0)$ will be a subgraph of $M^3(G_0)$ because $M(G_0) \subseteq M^2(G_0) \subseteq M^3(G_0)$. We then apply this logic iteratively to conclude that $|E(M^{k+1}(G_0))| - |E(M^k(G_0))| \geq 2$ whenever $k \geq 1$. Since the number of edges in the k^{th} matching graph of G_0 is a strictly increasing sequence of natural numbers, it must be unbounded. Hence, $|E(M^k(G_0))| \to \infty$ as $k \to \infty$.

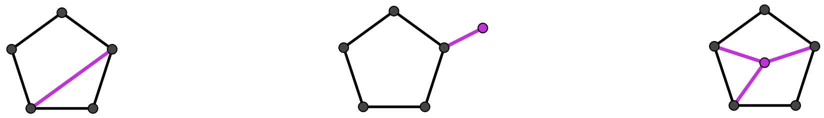


Figure 22: Graph G_0 and its first and second matching graphs

Now, suppose that G has N as a strict subgraph and consider the number of edges in $M(G)$. We know $|E(M(G))| \geq 6$ because $N \subseteq M(G)$. Now let $E' \subseteq E(G)$ be the set of edges in G which are not in the subgraph N. Every edge $e \in E'$ must be non-incident to at least two edges in $N \subseteq G$ because of the structure of N. See that we cannot place the purple edge in Figure 23 so that it is incident to more than four other edges. Hence, every $e \in E'$ will add at least 2 to the total edge count of $M(G)$. This means that

$$|E(M(G))| \geq 2|E'| + 6 = |E'| + |E(G)|.$$

If G has N as a *strict* subgraph, then $|E'| \geq 1$ and so we have $|E(M(G))| - |E(G)| \geq 1$. Now, we see that $M(G)$ must have N as a strict subgraph and we apply this logic again. Following this reasoning iteratively, we have $|E(M^{k+1}(G))| - |E(M^k(G))| \geq 1$ for all $k \geq 1$. So the number of edges in $M^k(G)$ is a strictly increasing sequence of natural numbers, implying that it must be unbounded. Hence, we have $|E(M^k(G))| \to \infty$ as $k \to \infty$.

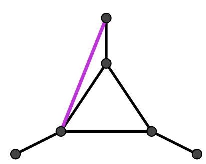


Figure 23

□

Corollary 5.3. *If G has C_5 or the net graph N as a snipped subgraph and $G \neq C_5$, $G \neq N$, then $|E(M^k(G))| \to \infty$ as $k \to \infty$.*

Proof. If G has C_5 as a snipped subgraph, then $M(G)$ will have C_5 as a subgraph by Lemma 2.11. Assuming that G differs from C_5 by more than isolated vertices, we know that $M(G) \neq M(C_5) = C_5$. Hence, we conclude that $M(G)$ has C_5 as a strict subgraph. Applying Lemma 5.2, we come to the desired conclusion. The same argument can be applied to N. □

Theorem 5.4 (Exploding Graph Theorem)**.** *If G has infinite d-value and G is not C_5 or the net graph N, then $|E(M^k(G))| \to \infty$ as $k \to \infty$.*

Proof. Suppose that G is a graph and $d(G) = \infty$. Then, by the Accumulation Theorem (5.1), we know that G accumulates C_5 or the net graph N. Since G is not either of these graphs, there must be some k such that $M^k(G)$ has C_5 or N as a strict subgraph. Hence, by Lemma 5.2, we know that $|E(M^k(G))| \to \infty$ as $k \to \infty$. □

Theorem 5.5. *$M^k(G) = G$ for some k iff $M(G) = G$ iff $G = C_5$ or $G = N$.*

Proof. (Direction 1) Suppose $M(G) = G$. We clearly then see that $M^k(G) = G$ for any k. Recall from [Aig69] that this occurs iff $G = C_5$ or $G = N$.

(Direction 2) Suppose that $M^K(G) = G$ for some positive integer K. For contradiction, assume that G is not C_5 or N. Because $M^K(G) = G$ we have $d(G) = \infty$. We then know that G will accumulate C_5 or N at some point by the Accumulation Theorem (5.1). By assumption, our graph is not C_5 or N so in fact G must accumulate C_5 or N as a *strict* subgraph. By Lemma 5.2, we see that the number of edges must increase without bound as we continue taking matching graphs. This is then a contradiction because under our starting assumptions, we must have $M^{nK}(G) = G$ for any positive integer n. We thus see that $G = C_5$ or $G = N$. For both of these graphs we know that $M(C_5) = C_5$ and $M(N) = N$. Hence, we must have $M(G) = G$. To finish, we just recall that $M(G) = G$ iff $G = C_5$ or $G = N$ from [Aig69]. □

Corollary 5.6. *If a graph G is not C_5 or the net graph, then there is no $k \geq 1$ such that $M^k(G) = G$.*

The above is an immediate corollary of Theorem 5.5.

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