Geometric Properties of Weighted Projective Space Simplices

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Collaborators

This work is joint with:

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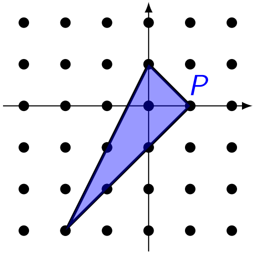
Brief Polytope Introduction

(Convex) polytopes are geometric objects that can be defined as the *convex hull* of finitely many points in $\mathbb{R}^n$.

Definition

A subset $P \subseteq \mathbb{R}^n$ is a *d-dimensional lattice polytope* if it is the convex hull of finitely many points in $\mathbb{Z}^n$ that span a d-dimensional affine subspace of $\mathbb{R}^n$.



$$P = \text{conv}\left\{(1,0),(0,1),(-2,-3)\right\}$$

Weighted Projective Space Simplices

Definition

Consider an integer partition $\mathbf{q} = (q_1, \ldots, q_d) \in \mathbb{Z}^d_{\geq 1}$ with the convention $q_1 \leq \cdots \leq q_d$. The lattice simplex associated with $\mathbf{q}$ is

$$\Delta_{(1,\mathbf{q})} := \operatorname{conv}\left\{\mathbf{e}_1, \ldots, \mathbf{e}_d, -\sum_{i=1}^{d} q_i \mathbf{e}_i\right\},$$

where $\mathbf{e}_i$ denotes the ith standard basis vector in $\mathbb{R}^d$.

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- $\mathcal{Q}$ = all lattice simplices of the form $\Delta_{(1,\mathbf{q})}$
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- Simplices in $\mathcal{Q}$ correspond to a subset of simplices defining weighted projective spaces:
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- Normalized Volume of $\Delta_{(1,\mathbf{q})}$: $1 + \sum_i q_i$

Useful Notation & Terminology

A natural parametrization on $\mathcal{Q}$ based on distinct entries:

- Suppose $\mathbf{q} = (q_1, \ldots, q_d)$ has t distinct entries given by $\mathbf{r} := (r_1, \ldots, r_t)$ with multiplicities $\mathbf{x} := (x_1, \ldots, x_t)$. Then, $\mathbf{q} = (q_1, \ldots, q_d) = (r_1^{x_1}, \ldots, r_t^{x_t})$.

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- We write $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ and say both $\mathbf{q}$ and $\Delta_{(1,\mathbf{q})}$ are *supported* by the vector $\mathbf{r}$ with *multiplicity* $\mathbf{x}$.

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- We write $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ and say both $\mathbf{q}$ and $\Delta_{(1,\mathbf{q})}$ are *supported* by the vector $\mathbf{r}$ with *multiplicity* $\mathbf{x}$.
- In this case, we say $\mathbf{q}$ and $\Delta_{(1,\mathbf{q})}$ are t-supported.

Two Important Properties

Definition

A lattice polytope P is reflexive if, possibly after translation by an integer vector, the origin is contained in P° and its geometric dual (or polar body)

$$P^* := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \in P\}$$

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A lattice polytope $P \subseteq \mathbb{R}^n$ has the integer decomposition property (or is IDP) if, for every integer $t \in \mathbb{Z}_{>0}$ and for all $\mathbf{p} \in tP \cap \mathbb{Z}^n$, there exists $\mathbf{p}_1, \ldots, \mathbf{p}_t \in P \cap \mathbb{Z}^n$ such that $\mathbf{p} = \mathbf{p}_1 + \cdots + \mathbf{p}_t$.

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We will be interested in $\Delta_{(1,\mathbf{q})}$ that are simultaneously IDP and reflexive.

Motivation: Why IDP and Reflexive?

Conjectures in commutative algebra have motivated a significant amount of study on the Ehrhart theory of lattice polytopes.

- Ehrhart theory: lattice point enumeration of dilates of polytopes
- What geometric properties of P are necessary/sufficient for $h^*(P; z)$ to be symmetric, unimodal, etc?

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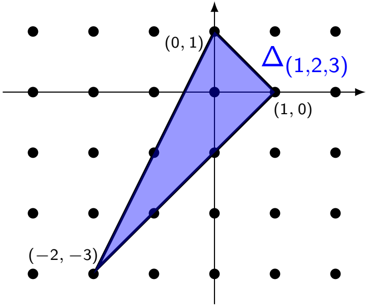
Lattice simplices have been shown to form a rich testing ground of examples with which to vet this conjecture.

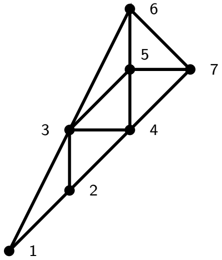
- Leverage algebraic/geometric properties of special classes

Project I: Triangulations of 2-Supported $\Delta_{(1,\mathbf{q})}$

Definition

Let $\mathbf{A} \subseteq \mathbb{R}^d$ be a point configuration. A *triangulation* $\mathcal{T}$ of $\mathbf{A}$ is unimodular if every simplex has normalized volume one. We say $\mathcal{T}$ is *regular* if it can be obtained by projecting the lower envelope of a lifting of $\mathbf{A}$ from $\mathbb{R}^{d+1}$.





Motivation: 2-Supported $\Delta_{(1,\mathbf{q})}$ Triangulations

Theorem

If a lattice polytope P admits a unimodular triangulation, then P is IDP.

Theorem (Braun-Davis-Solus, 2016)

If $\Delta_{(1,\mathbf{q})}$ is 2-supported and IDP reflexive, then $\Delta_{(1,\mathbf{q})}$ has a unimodal Ehrhart h^*-vector.

Theorem (Bruns-Römer, 2007)

If P is reflexive and admits a regular unimodular triangulation, then P has a unimodal Ehrhart h^*-vector.

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Thus, it is of interest to determine if IDP reflexive lattice polytopes admit regular unimodular triangulations.

2-Supported $\Delta_{(1,\mathbf{q})}$ (cont.)

Theorem (Braun-Davis-Solus, 2016)

Suppose $\mathbf{q}$ *is 2-supported. The simplex* $\Delta_{(1,\mathbf{q})}$ *is IDP reflexive if and only if* $\mathbf{q}$ *is of the form* $(r_1^{x_1}, r_2^{x_2})$ *where either*

1. $r_1 = 1$ *with* $r_2 = 1 + x_1$ *and* x_2 *arbitrary, or*
2. $r_1 > 1$ *with* $r_2 = 1 + r_1 x_1$ *and* $x_2 = r_1 - 1$.

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- Goal: establish a regular unimodular triangulation of the point configuration formed by the lattice points of $\Delta_{(1,\mathbf{q})}$ in case (2).

Lattice Point Characterization

Define $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_{r_1+3}, \mathbf{b}_1, \ldots, \mathbf{b}_d\} \subset \mathbb{Z}^d$, where

$$\begin{aligned} \mathbf{a}_{r_1+1} &= ((-1)^{x_1}, (-x_1)^{r_1-1}), \\ \mathbf{a}_{r_1+2} &= (0^{x_1}, (-1)^{r_1-1}), \\ \mathbf{a}_{r_1+3} &= (0^{x_1}, 0^{r_1-1}), \\ \mathbf{a}_i &= (r_1 - i + 1)\mathbf{a}_{r_1+1} + \mathbf{a}_{r_1+2} \text{ for } 1 \leq i \leq r_1 \\ \mathbf{b}_j &= \mathbf{e}_{d-j+1} \text{ for } 1 \leq j \leq d \end{aligned}$$

Example:
For $\mathbf{q} = (2, 3)$,
$\mathcal{A} = \{(-2, -3), (-1, -2), (-1, -1), (0, -1), (0, 0), (0, 1), (1, 0)\}$

Theorem (Braun-H., 2020)

The lattice points of the 2-supported IDP simplex $\Delta_{(1,\mathbf{q})}$ are given by $\mathcal{A}$.

2-Supported IDP Reflexive $\Delta_{(1,\mathbf{q})}$ Admit a Regular Unimodular Triangulation

Theorem (Braun-H., 2020)

There exists a lexicographic squarefree initial ideal of the toric ideal associated with $\mathcal{A}$.

Corollary

Assume $\mathbf{q} = (r_1^{x_1}, (1 + r_1 x_1)^{r_1 - 1})$ with $r_1 > 1$. The convex polytope $\Delta_{(1,\mathbf{q})} = \mathrm{conv}\,\{\mathcal{A}\}$ admits a regular unimodular triangulation induced by the lexicographic term order $<_{lex}$.

Project II: 3-Supported $\Delta_{(1,\mathbf{q})}$

- Purpose: Extend known results for 1- and 2-supported $\Delta_{(1,\mathbf{q})}$ to the 3-supported case
  - Provide a complete characterization of 3-supported IDP reflexive $\Delta_{(1,\mathbf{q})}$
  - Explore Ehrhart unimodality

Foundational IDP Result

Theorem (Braun-Davis-Solus, 2016)

The reflexive simplex $\Delta_{(1,\mathbf{q})}$ *is IDP if and only if for every* $j = 1, \ldots, n$ *and for all* $b = 1, \ldots, q_j - 1$ *satisfying*

$$b\left(\frac{1+\sum_{i\neq j} q_i}{q_j}\right) - \sum_{i\neq j}\left\lfloor\frac{bq_i}{q_j}\right\rfloor \geq 2,$$

there exists a positive integer $c < b$ *satisfying the following equations:*

$$\left\lfloor\frac{bq_i}{q_j}\right\rfloor - \left\lfloor\frac{cq_i}{q_j}\right\rfloor = \left\lfloor\frac{(b-c)q_i}{q_j}\right\rfloor, \text{ and}$$

$$c\left(\frac{1+\sum_{i\neq j} q_i}{q_j}\right) - \sum_{i\neq j}\left\lfloor\frac{cq_i}{q_j}\right\rfloor = 1,$$

where the first is considered for all $1 \leq i \leq n$ *with* $i \neq j$.

IDP Result (cont.)

Corollary

If $\Delta_{(1,\mathbf{q})}$ is reflexive and IDP, then for all $j = 1, 2, \ldots, n$,

$$1 + \sum_{i=1}^{n} (q_i \bmod q_j) = q_j$$

or equivalently

$$1 + \sum_{i=1}^{n} x_i (r_i \bmod r_j) = r_j.$$

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Any $\mathbf{q}$ satisfying these equations for all j is said to satisfy *the necessary condition for IDP*.

Stratify by Multiplicity Instead of Support

Theorem (Braun-Davis-H.-Lane-Solus, 2021)

Consider a 3-supported vector $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ *such that* $\Delta_{(1,\mathbf{q})}$ *satisfies the necessary condition for IDP. If* $\mathbf{x} = (x_1, x_2, x_3)$*, then* $\mathbf{r}$ *is of one of the following forms:*

(i) $\mathbf{r} = (1, 1 + x_1, (1 + x_1)(1 + x_2))$.

(ii) $\mathbf{r} = (1 + x_2, 1 + x_1(1 + x_2), (1 + x_1(1 + x_2))(1 + x_2))$.

(iii) $\mathbf{r} = ((1 + x_2)(1 + x_3), 1 + x_1(1 + x_2)(1 + x_3), (1 + x_1(1 + x_2)(1 + x_3))(1 + x_2))$.

(iv) $\mathbf{r} = (1, (1 + x_1)(1 + x_3), (1 + x_1)(1 + x_2(1 + x_3)))$.

(v) $\mathbf{r} = (1+(1+x_3)x_2, (1+x_3)(1+x_1(1+(1+x_3)x_2)), (1+(1+(1+x_3)x_2)x_1)(1+(1+x_3)x_2))$.

(vi) $\mathbf{r} = ((1 + x_3)(1 + (1 + x_3)x_2), (1 + x_3)(1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)), (1 + (1 + x_3)(1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2))$.

(vii) $\mathbf{r} = (1 + x_3, (1 + x_3)(1 + x_1(1 + x_3)), (1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2))$.

(viii) *There exists some* $k, s \geq 1$*, where*

$$\mathbf{r} = (1 + kx_2, (skx_2 + s + k)(1 + x_1(1 + kx_2)), (1 + x_1(1 + kx_2))(1 + x_2(skx_2 + s + k))),$$
$$\mathbf{x} = (x_1, x_2, skx_2 + s - k + 1)\,.$$

Further, the first seven $\mathbf{r}$*-vectors produce IDP* $\Delta_{(1,\mathbf{q})}$*'s, while* (*viii*) *does not.*

Unimodality in the IDP Reflexive Case

Theorem

If $\mathbf{q}$ *is a 1- or 2-supported vector yielding an IDP reflexive* $\Delta_{(1,\mathbf{q})}$*, then* $\Delta_{(1,\mathbf{q})}$ *is* h^**-unimodal.*

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For $\mathbf{x} = (x_1, x_2, x_3)$ *a positive integer vector, if* $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ *where* $\mathbf{r}$ *is one of the following forms, then* $\Delta_{(1,\mathbf{q})}$ *is* h^**-unimodal.*

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(iv) $\mathbf{r} = (1, (1 + x_1)(1 + x_3), (1 + x_1)(1 + x_2(1 + x_3)))$.

Experimental evidence suggests the remaining four cases are also h^*-unimodal.

Questions?

2-Supported $\Delta_{(1,\mathbf{q})}$ Preprint: `https://arxiv.org/abs/2010.13720`
3-Supported $\Delta_{(1,\mathbf{q})}$ Preprint: `https://arxiv.org/abs/2103.17156`