Enumerating simplicial spanning trees of shifted and color-shifted complexes, using simplicial effective resistance

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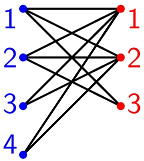
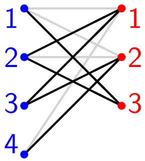
Ferrers graphs (Ehrenborg-van Willigenburg '04)

Example ($\langle 42, 23 \rangle$)

	1	2	3	4
1	11	21	31	41
2	12	22	32	42
3	13	23		



Spanning trees of Ferrers graphs

- $T =$ 

$\text{wt } T = (1234)(123)23123$

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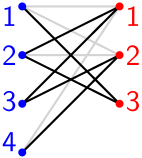
- $T =$ 

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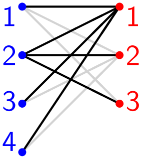
- $T =$ 

wt $T = (1234)(123)2^2 1^3$

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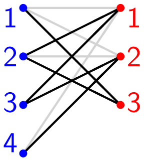
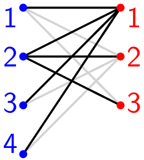
- $T =$ 

 wt $T = ({\color{blue}1234})({\color{red}123}){\color{blue}23}{\color{red}123}$
- $T =$ 

 wt $T = ({\color{blue}1234})({\color{red}123}){\color{blue}2}^2{\color{red}1}^3$
- Total is $({\color{blue}1234})({\color{red}123})({\color{blue}1} + {\color{blue}2} + {\color{blue}3} + {\color{blue}4})({\color{blue}1} + {\color{blue}2})({\color{red}1} + {\color{red}2} + {\color{red}3})({\color{red}1} + {\color{red}2})^2$

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Theorem (Ehrenborg-van Willigenburg)

This works in general

Proof – by electrical network theory!

- Set $I_{ij} = 1$
- Set $R_{pq} = (pq)^{-1}$
- Find remaining currents so they satisfy Kirchhoff's Laws
- Compute V_{ij}, which is effective resistance since $I_{ij} = 1$

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Theorem (Thomassen '90)

$$V_{ij} = \frac{\textit{spanning trees with } ij}{\textit{spanning trees without } ij}$$

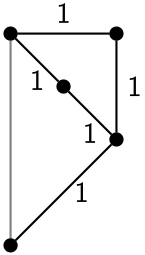
From this, we can easily get

$$\frac{\text{spanning trees of (graph with } ij)}{\text{spanning trees of (graph without } ij)}$$

Now apply induction

Example (Unweighted)

Example ($K_{3,2} = \langle 32 \rangle$)



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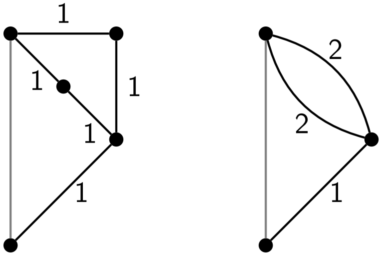

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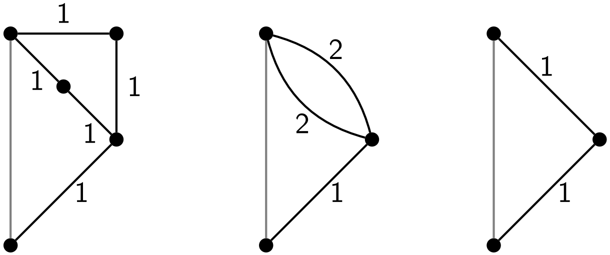

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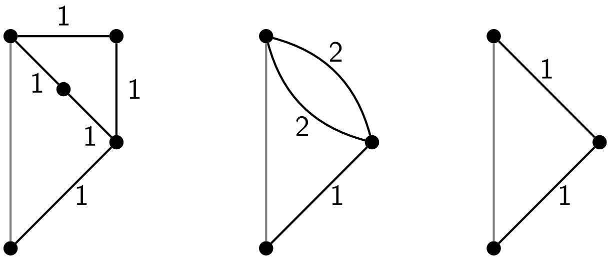

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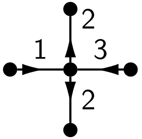
$$\frac{\text{trees with edge}}{\text{trees without edge}} = \frac{8}{4} = 2$$

Kirchhoff's Laws

Start with a simple graph. Each edge has a positive resistance R, directed current I, and directed voltage drop V

Kirchhoff's Laws



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- Sum of currents at a vertex is 0
- $\ker \partial_1$
- spanned by directed cycles

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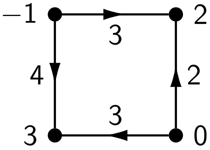

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Can "solve" circuits by minimizing energy (RI^2 on each edge)

Simplicial networks (Catanzaro-Chernyak-Klein '15)

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We still have energy minimization.

Simplicial effective resistance

Let σ be a facet of simplicial complex X

- Set $I_\sigma = 1$
- Set $R_\tau = (x_\tau)^{-1}$ for all other facets τ.
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Theorem (Kook-Lee '18)

$$V_\sigma = \frac{\hat{k}_d(X)_\sigma}{\hat{k}_d(X - \sigma)}$$

where $\hat{k}_d$ is a torsion-weighted simplicial tree count, and $\hat{k}_d(X)_\sigma$ means restricted to trees containing σ.

Simplicial spanning trees (Kalai '83; D.-Klivans-Martin '09)

Let X be a d-dimensional simplicial complex.
$T \subseteq X$ is a **simplicial spanning tree** of Δ when:

0. $T_{(d-1)} = X_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(T;\mathbb{Z})$ is a finite group ("connected");
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3. $f_d(T) = f_d(X) - \tilde{\beta}_d(X) + \tilde{\beta}_{d-1}(X)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third.
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$$k_d(X) = \sum_{T \in \mathcal{T}(X)} |\tilde{H}_{d-1}(T,\mathbb{Z})|^2$$

$$\hat{k}_d(X) = \sum_{T \in \mathcal{T}(X)} |\tilde{H}_{d-1}(T,\mathbb{Z})|^2 \operatorname{wt} T$$

Color-shifted complexes

Definition (Babson-Novik, '06)

A color-shifted complex is a simplicial complex with:

- vertex set $V_1 \dot{\cup} \dots \dot{\cup} V_r$ (V_i is set of vertices of color i);
- every facet contains one vertex of each color; and
- if $v < w$ are vertices of the same color, then you can always replace w by v.

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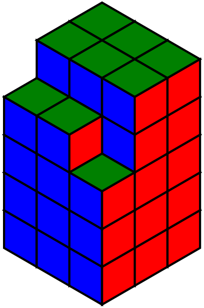
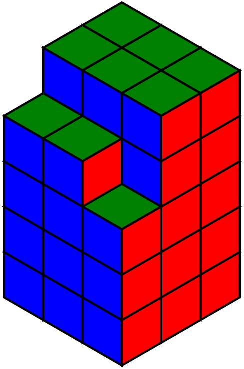
Example

$\langle 235, 324, 333 \rangle$



Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$$(1^7 2^7 3^6)(1^7 2^7 3^7)(1^5 2^5 3^5 4^5 5^4)$$
$$\times (1+2+3)^5 (1+2)^3 \ (1+2+3)^6 (1+2)$$
$$\times (1+\cdots+5)^2 (1+2+3+4)(1+2+3)$$



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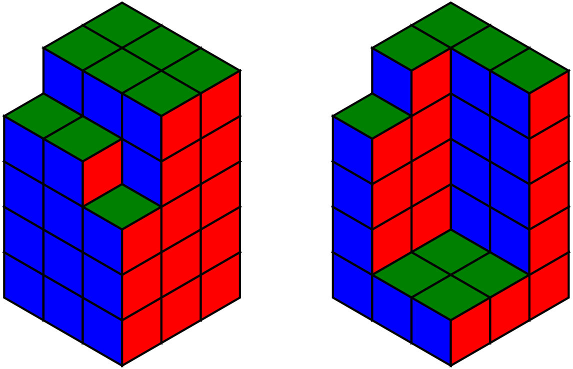

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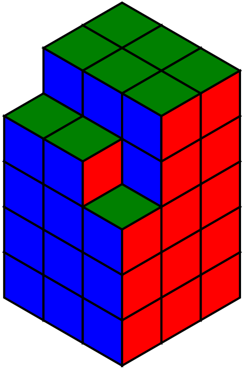
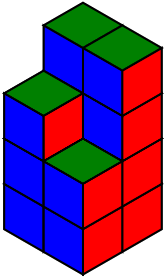
$$\times (1+2+3)^5 (1+2)^3$$





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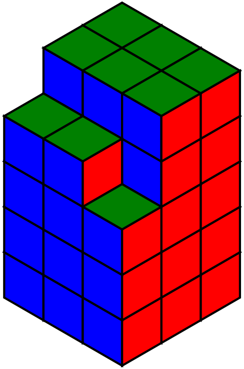
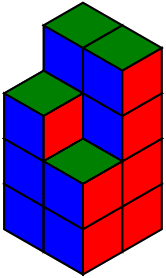
$\times$

$(1+2+3)^6(1+2)$





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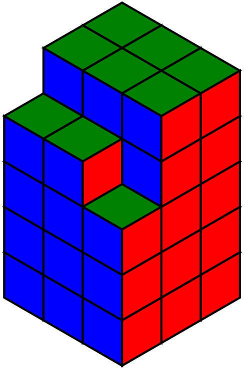
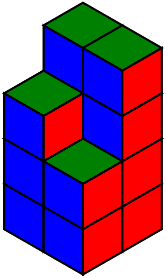
$$\times\, (1 + \cdots + 5)^2 (1 + 2 + 3 + 4)(1 + 2 + 3)$$





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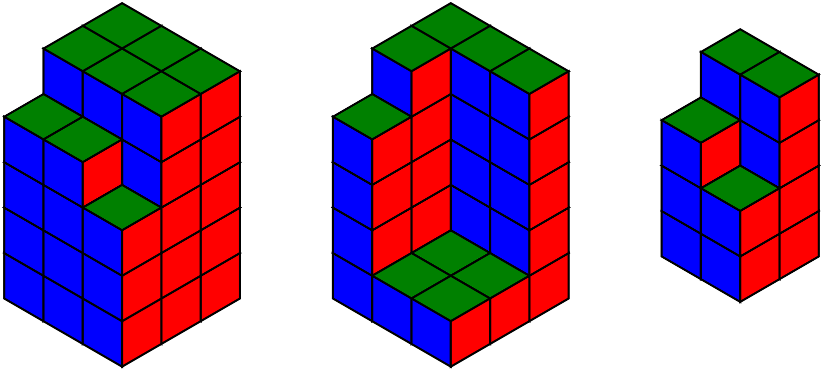
$$
\begin{aligned}
&(1^7 2^7 3^6)(1^7 2^7 3^7)(1^5 2^5 3^5 4^5 5^4) \\
&\quad \times (1+2+3)^5 (1+2)^3 \; (1+2+3)^6 (1+2) \\
&\quad \times (1+\cdots+5)^2 (1+2+3+4)(1+2+3)
\end{aligned}
$$



Proof

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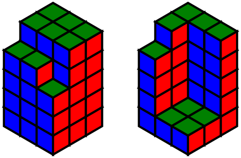

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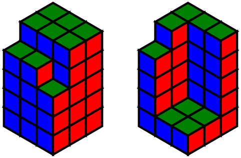

Conjectured by Aalipour-AD (long matrix manipulation pf. $r = 3$)

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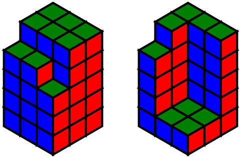

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Proof by simplicial effective resistance (DKLM):

- $(1^7 2^7 3^6)(1^7 2^7 3^7)(1^5 2^5 3^5 4^5 5^4)(1^8 1^7 1^4)$ for initial tree

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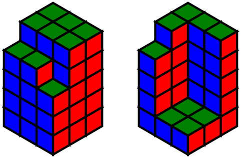

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Proof by simplicial effective resistance (DKLM):

- $(1^7 2^7 3^6)(1^7 2^7 3^7)(1^5 2^5 3^5 4^5 5^4)(1^8 1^7 1^4)$ for initial tree
- induction (ex.) When adding in 235, effective resistance says

$$\frac{\text{trees in new complex}}{\text{trees in original complex}} = \frac{1+2}{1} \frac{1+2+3}{1+2} \frac{1+\cdots+5}{1+\cdots+4}$$

Shifted complexes

Definition

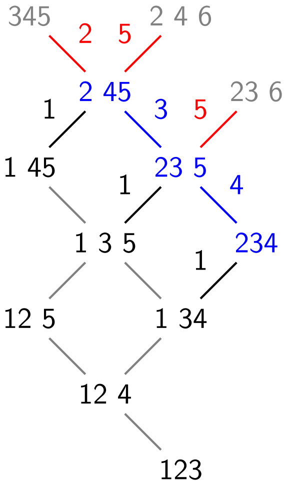
A shifted complex is a simplicial complex with:

- vertex set $1, \ldots, n$;
- if $v < w$, then you can always replace w by v.

Example ($\langle 245 \rangle$)

$123, 124, 125, 134, 135, 145, 234, 235, 245$

Enumerating spanning trees of shifted complexes



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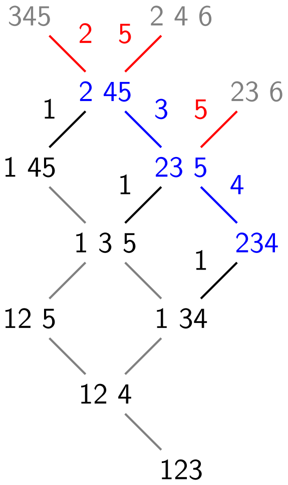

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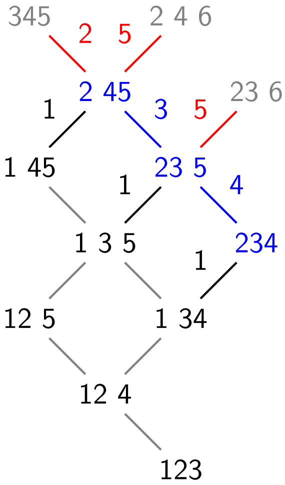

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- Start with spanning tree of facets with 1
- When adding (e.g.) 23 5, effective resistance says

$$\frac{D_2 D_3 \; D_5}{D_1 D_2 \; D_4} = \frac{D_3}{D_1} \frac{D_5}{D_4}$$

where $D_j = x_1 + \cdots + x_j$.

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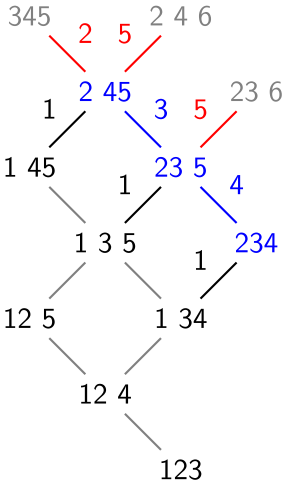

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  where $D_j = x_1 + \cdots + x_j$.
- When done, left with red edges divided by black edges with 1's.