1. Let $T\ V \to V$. Prove the following:

a) If V is finite dimensional then
T is invertible iff 0 is not an eigenvalue of T.

b) Suppose T is invertible. Then λ is an eigenvalue of T iff $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} and $E_\lambda(T) = E_{\lambda^{-1}}(T^{-1})$

c) If λ is eigenvalue for T with eigenvector v then λ^m is eigenvalue for T^m with eigenvector v and $E_\lambda(T) \subseteq E_{\lambda^m}(T^m)$

d) Give an example of $T\ V \to V$ s.t λ is eigenvalue for T^m but T has no eigenvalue ℓ s.t $\ell^m = \lambda$

e) Give an example of $T\ V \to V$ such that λ is an eigenvalue for T and $E_\lambda(T) \neq E_{\lambda^m}(T^m)$

f) Find $T\ V \to V$ that does not have 0 as eigenvalue but is not invertible.

(2) Suppose $S, T \in \mathcal{L}(V)$ and S is invertible.
Prove that T and $S^{-1}TS$ have the same eigenvalues. What is the relationship between the eigenvectors of T and those of $S^{-1}TS$?

(3) Find all eigenvalues and eigenvectors of
$T \quad C^{\infty} \to \mathbb{R}^{\infty}$
$$T(x_1, x_2 \cdots) = (x_2, x_3 \cdots)$$

(4) Suppose $\lambda_1, \lambda_2, \ldots \lambda_n$ are distinct real numbers
Prove $e^{\lambda_1 x}, e^{\lambda_2 x} \ldots e^{\lambda_n x}$ are linearly independent vectors in $C^{\infty}(R)$.

(5) Let $T \in \mathcal{L}(V)$, $\lambda \in F$, $n \geq 1$
Show that $T(T-\lambda I)^n = (T-\lambda I)^n T$

(6) Consider $T \in \mathcal{L}(V)$ $\dim V = n$, and a vector v in V
s.t $(T-\lambda I)^{p-1} v \neq 0$ $(T-\lambda I)^p v = 0$. Let $v_j = (T-\lambda I)^j v$
for $j = 1, \dots p-1$, $v_0 = v$
Show that the vectors $v_{p-1}, \dots, v_1, v_0$ are linearly independent (Hint: start assuming $a_0 v_0 + \dots + a_{p-1} v_{p-1} = 0$
and apply $(T-\lambda I)^{p-1}$ to both sides of this equation)
Show $\operatorname{span}(v_0, \dots v_{p-1})$ is T invariant.
(Hint: for any w in V we can write $Iw = (T-\lambda I)w + \lambda w$

(7) Prove that if $P \in \mathcal{L}(V)$ and $P^2 = P$
then $V = N(P) + R(P)$ ($N(P)$ = null space of P
$R(P)$ = range(P)). In general, is the sum direct?