Lesson 9

Linear transformations

Null space , Range

Textbook: 2.1

Recall given a lineer transformation
$T \quad V \longrightarrow W$

$N(T) = \{ v \in V \ / \ T(v) = 0 \}$

$R(T) = \{ w \in W \ / \ \exists v \in V \quad T(v) = w \}$

Def $\quad \dim(N(T)) = $ nullity of T
$\qquad \dim(R(T)) = $ renk of T

when $N(T)$ end $R(T)$ are finite dimensionel.

Th (dimension th)
Given $T \quad V \to W$, if V is finite dimensionel then

$\dim V = \text{nullity}(T) + \text{renk}(T)$

Ex: $T \quad \mathbb{R}^n \to \mathbb{R}^m$

$$T(x_1, \dots x_n) = \underset{m \times n}{M} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

nullity of T = dim $N(T)$

= nullity of M

renk of T = dim $R(T)$

= renk M

n = nullity (M) + renk (M)

In a system with n veriebles

free veriables + # dependent veriebles = n

.)

Recall : a function $f \; X \to Y$
is injective (1-1) if $\forall x_1, x_2 \in X$
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
(or $\forall x_1, x_2 \in R \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$)

Th $T \; V \to W$
is injective iff $N(T) = \{0\}$ i.e if
$T(v) = T(0) \Rightarrow v = 0$

Proof :

Recall : a function $f \; X \to Y$ is surjective (onto)
if $\forall y \in Y \; \exists x \in X \;\; y = f(x)$
that is $Y = \text{Range}(f)$.

Therefore: $T \; V \to W$ is surjective iff
$R(T) = W$

Th: Assume V and W are finite dimensional vector spaces with $\dim V = \dim W$.
Let $T: V \to W$ be a linear transformation then the following are equivalent

1) T is one to one
2) T is onto
3) $\operatorname{rank}(T) = \dim V$

The situation is different for infinite dim vector spaces

Ex $T \quad P(\mathbb{R}) \to P(\mathbb{R})$

$T(p) = p \cdot x$

check it is a lineer transformation

$N(T) =$

$R(T) =$

Ex $\quad T \; \mathbb{R}^{\infty} \longrightarrow \mathbb{R}^{\infty}$

$T(x_1, x_2 \cdots\cdots) = x_2, x_3 \cdots\cdots$

$N(T) = \{ x, 0, 0 \cdots\cdots \mid x \in \mathbb{R} \}$

$R(T) = \mathbb{R}^{\infty}$