Lesson 8

Linear transformations

Null space , Range

Textbook: 2.1

Def: given two vector spaces V and W over F

A function $T\ V \to W$ is a linear transformation iff

$T(\lambda v) = \lambda T(v) \quad \forall \lambda \in F, \forall v \in V$

$T(v+w) = T(v) + T(w) \quad \forall v, w \in V$

Ex $T\ P(\mathbb{R}) \to P(\mathbb{R})$

$T(p) = p'$

is a linear transformation since

Ex $T \quad F^{\infty} \to F^{\infty}$

$T(x_1, x_2, x_3 \cdots) = (x_2, x_3 \cdots \quad)$

Is a linear transformation since

$T(k\{x_n\}) = T(\{kx_n\}) = (kx_2, kx_3, kx_4 \cdots) = k\,T(\{x_n\})$

$T(\{x_n\} + \{y_n\}) = T(\{x_n + y_n\}) = (x_2+y_2, x_3+y_3 \cdots) = (x_2, x_3 \cdots)$

$+ (y_2, y_3, \cdots\cdots) = T(\{x_n\}) + T(\{y_n\})$

Ex $T\ R^3 \to R^3$ projection on y, z plane

$T(x_1, x_2, x_3) = (0, x_2, x_3)$

is a linear transformation

Ex $T\ F^\infty \to F^\infty$

$T((x_1, x_2, x_n \dots) = (0, x_1, x_2, x_n \dots \dots \dots)$ is a linear transformation

Ex $T \quad \mathbb{R}^2 \to \mathbb{R}^2$ reflection

$T((x,y)) = (-x, y)$

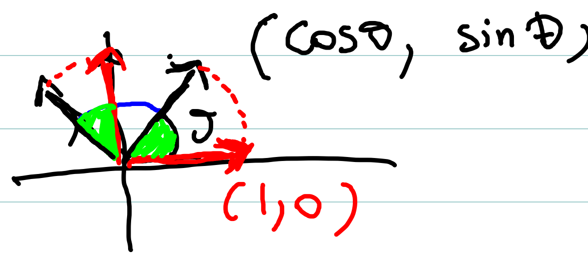
Ex: $T \quad \mathbb{R}^2 \to \mathbb{R}^2$ rotation

$$T((x,y)) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$T((1,0)) = (\cos\theta, \sin\theta)$

$T((0,1)) = (-\sin\theta, \cos\theta)$

$(-\sin\theta, \cos\theta)$



$T \quad R^n \to R^m$

$T(v) = A v$ where $A \in M_{mn}(R)$

Is a lineer transformation since

$T(kv) = A(kv) = k(Av) = k\,T(v)$

$T(v+w) = A(v+w) = Av + Aw = T(v) + T(w)$

for all vectors $v, w \in R^n$ and scalers $k \in R$

Ex $T\ R^3 \to R^3$

$T(x\,y\,z) = (x+1, 2y, z+x-y)$ NOT A LINEAR TRANSFORMATION

Th If $T\ V \to W$ is a linear transformation then $T(0) = 0$

Def : Given T $V \to W$ The kernel of T (or null space) $N(T)$ is the set $\{v \in V / T(v) = 0\}$. The range of T, denoted $R(T)$ is the set $\{w \in W / w = T(v)$ for some v in $V\}$

Th : $N(T) \leq V$, $R(T) \leq W$

Th If V, W are vector spaces and $B = \{v_1, v_2, \dots v_n\}$ is a basis for V and $T \; V \to W$ is a linear transformation then $R(T) = \text{Span}(T(v_1), \dots, T(v_n))$

Ex What are the null space and Range of

$T \quad P(\mathbb{R}) \to P(\mathbb{R})$

$T(p) = p'$

Ex $T \quad F^{\infty} \to F^{\infty}$

$T(x_1, x_2 \cdots x_n \cdots) = (x_2, x_3, \cdots\cdots)$

Quiz T F $T\ V \to W$

$v_1 \cdots v_m$ lin indep then

$T(v_1)\ T(v_2) \cdots T(v_n)$ lin indep

F example $T\ R^2 \to R^2$

$T(x\ y) = (x\ 0)$

$T(v_1)\ T(v_2) \cdots T(v_n)$ lin indep

$v_1\ v_2 \cdots v_n$ lin indep

True $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$

$T(\ )\ (\ )\ (\ ) = 0$