Lesson 7

Infinite dimensional

Textbook 1.7

Important facts about finite dimensional spaces

1. They have bases
2. Any independent set can be extended to a basis
3. Any spanning set contains a basis
4. All bases have the same number of elements

Examples $P(R)$

No finite bases, but $B = \{1, x, x^2, \dots\}$ is a basis

Example R^∞

$S = \{e_1, e_2, e_3, \dots e_n, \dots\}$

$e_i = 0, 0, \dots 0, \underset{i}{1}, 0, \dots$

Is S linearly independent?

Does it span R^∞

Axiom of choice and infinite dimensional vector spaces:

Axiom of Choice: given any family $\mathcal{F}$ of nonempty sets ($\mathcal{F} = \{ S_x \mid x \in X \}$) there is a function $f \quad \mathcal{F} \to \bigcup_{x \in X} S_x$ st $f(S) \in S$

Maximal priciple : Let $\mathcal{F}$ be a family of sets If for every chain $C \cdots S_x \subseteq S_y \subseteq S_z \cdots$ of sets in $\mathcal{F}$ there exists a set in $\mathcal{F}$ that contains every set in C then $\mathcal{F}$ contains a maximal element, that is a set S that is not properly contained in any set of $\mathcal{F}$

Th: The axiom of choice and maximal principle are equivalent.

Th Let S be a linearly independent subset of a vector space V, then there exists a maximal linearly independent subset of V that contains S. Such maximal linearly independent set B is a basis for V.

Compare with proof for finite dimensional case:

$S = S_1 = v_1, v_2, \dots v_k$ Keep adding vectors until you have a basis.

$$S = S_1 \subseteq S_2 \subseteq \dots \subseteq S_m = B_1$$
$$\subseteq T_2 \subseteq \dots \subseteq T_m = B_2$$

Infinite dimensional

$$S = S_1 \subseteq S_2 \subseteq \dots \subseteq S_n \subseteq \dots \subseteq S_\omega \subseteq S_{\omega+1} \subseteq \dots$$

What maximal principle says is that this process must terminate

Th Let V be a vector space and let S span V then S contains a basis for V