Lesson 5

Bases and dimension

Textbook: 1.6

Def : $B \subseteq V$ is a basis for V if
B is linearly independent and spans V

Note : We say $\emptyset$ is a basis for $V = \{0\}$

Th : Let $S = \{v_1, \dots, v_n\}$ span $V \neq \emptyset$
then S contains a basis for V

Proof: Consider the list $v_1 \dots v_n$
Construct B as follows

Th: Every finitely generated vector space has a finite basis.

Th: If V is finitely generated any finite independent set of vectors $v_1 \cdots v_k$ can be extended to a basis $v_1 \cdots v_k\ w_1 \cdots w_m$.

If V is finitely generated

Th 1: Any finite spanning set S for V contains a basis for V

Th 2 Any finite independent set $S \subseteq V$ is contained in a basis of V.

Th (Replacement): Let V be a vector space, $V = \text{span}\{v_1, v_2, \dots v_n\}$ and let $\{w_1, \dots w_m\} \subseteq V$ be linearly independent. Then $m \le n$ and there exists $n-m$ vectors $v_{i_1} \dots v_{i_{n-m}}$ in $\{v_1 \dots v_n\}$ such that the set $S = \{w_1 \dots w_m\} \cup \{v_{i_1}, \dots v_{i_{n-m}}\}$ spans V.

Proof: By induction on m

Th If V has one finite basis all of its bases have the same number of elements

Proof:

Def: If V has a finite basis with n elements we say the dimension of V is n and we write $\dim V = n$.

Note: $\dim \{0\} = 0$

Th: $S = \{v_1, v_2, \dots, v_n\}$ is a basis for V iff each vector in V can be expressed in a unique way as linear combination of vectors of S.

Proof: Assume S is a basis for V, then every $v \in V$ can be written as a linear combination $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. Assume that $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$ as well. Then

$$(\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \cdots + (\alpha_n - \beta_n) v_n = 0$$

since S is independent $\alpha_i = \beta_i$ for all i

Now assume each vector can be written as a linear combination of elements of S, in a unique way then S spans V and 0 can only be written as $0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$ so S is independent.

Def If $B = \{b_1, b_2, \dots b_n\}$ is a basis for V and $v \in V$ $v = k_1 b_1 + \cdots + k_n b_n$ we will denote $(k_1, k_2, \dots k_n)$ by $[v]_B$ and call this vector the vector of coordinates of v with respect to B.

Ex 1 Dim $R^n = n$

Basis $(1\,0\,0 \cdots 0)$ $(0\,1\,0 \cdots 0)$ $\cdots$ $(0 \cdots 0\,1)$

Ex: Dim $C^n = n$

Basis $(1\,0\,0 \cdots 0)$ $(0\,1\,0 \cdots 0)$ $\cdots$ $(0 \cdots 0\,1)$

Ex Dim $M_{m \times n}(F) = m \cdot n$

Basis $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \\ 0 & \cdots & & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & & & 0 \\ 0 & \cdots & & & 0 \end{bmatrix}$ $\cdots$

$\cdots$ $\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$

Ex $P_n(\mathbb{R})$ has dimension $n+1$

$\{1, x, x^2, \dots x^n\}$ is a basis

Ex $P(\mathbb{R})$ is not finite dimensional

$B = \{x^n \mid n \in \mathbb{Z}, n \geq 0\} = \{1, x, x^2, \dots\}$

is a basis for $P(\mathbb{R})$

Th Let $\dim V = n$ then

1) Any spanning subset S of V with n elements is a basis.
2) Any independent subset T of V with n elements is a basis.

Proof:

1) S must contain a subset B which is a basis for V, so B has n elements therefore $B = S$
2) T can be extended to a basis B for V with n elements so $T = B$

Note: If $\dim(V) = n$ any independent subset has $\leq n$ elements, any spanning set has $\geq n$ elements