Lesson 4

Spanning sets, linearly independent sets

Textbook: 1.4, 1.5

Def: Given $v_1, \dots, v_m \in V$, $\alpha_1, \alpha_2, \dots \alpha_m \in F$

$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$ is called a linear combination of $v_1, \dots, v_m$.

Def: Given $S \subseteq V$ Span(S) is the set containing all linear combinations of vectors of S.

By convention $\text{Span}(\phi) = \{0\}$

Ex $\text{Span}((1\,0)) = \{(x, 0)\}$

Ex $\text{Span}(\{(1\,0\,0), (0\,1\,0)\}) = \{(\alpha_1, \alpha_2, 0)\}$

Ex: $V = P(\mathbb{R})$. $\text{Span}(\{x+1, x-1\})$ consists of

Th: If $S \subseteq V$, Span(S) is the smallest subspea of V containing S

Proof:

Def: Given $S \subseteq V$ we say S generates (or spans) V if $V = \text{span}(S)$

Def A vector space V is finitely generated if there is a finite $S \subseteq V$ that spans it.

Ex $P(\mathbb{R})$ is not finitely generated

Def: $v_1, \dots, v_m \in V$ are linearly independent if $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0$ implies $\alpha_1 = \alpha_2 \dots = \alpha_m = 0$. vectors that are not linearly independent are called linearly dependent.

Def $S \subseteq V$ is linearly independent if for every finite list $v_1, \dots v_m$ of vectors in S $v_1, \dots v_m$ are linearly independent.

Ex: $\{0\}$ is linearly dependent

Ex:

Are
$$v_1 = (1, 1, -1, 0)$$
$$v_2 = (2, 3, 0, 1)$$
$$v_3 = (1, 2, -3, -1)$$
$$v_4 = (1, 1, 1, 1)$$

linearly independent?

Consider

$$\lambda_1(1,1,-1,0) + \lambda_2(2,3,0,1) + \lambda_3(1,2,-3,-1) +$$
$$+ \lambda_4(1,1,1,1) = (0,0,0,0) \quad ?$$

The vectors are linearly independent iff the system below has only the trivial solution:

$$\begin{aligned}
\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_4 &= 0 \\
\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 &= 0 \\
-\lambda_1 \quad\quad - 3\lambda_3 + \lambda_4 &= 0 \\
\lambda_2 - \lambda_3 + \lambda_4 &= 0
\end{aligned}$$

or

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ -1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ -1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

want this system to only have one solution $(0, 0, 0, 0)$

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are free variables, system has infinitely many solutions, so v_1, v_2, v_3, v_4 are not independent. Pivot columns are independent: $(1,1,-1,0)$ $(2,3,0,1)$ $(1,1,1,1)$ and $\text{span}(v_1, v_2, v_3, v_4) = \text{span}(v_1, v_2, v_4)$

Proof: If we remove the columns that correspond to free variables we obtain a system that has only the 0 solution.

The span is the same since I can find sol

$x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = 0$ with $x_3 \neq 0$

so $v_3 = -\frac{x_1}{x_3} v_1 - \frac{x_2}{x_3} v_2 - \frac{x_4}{x_3} v_4$

Th: Let S be a linearly independent subset of V and let $v \in V$, $v \notin S$ then $S \cup \{v\}$ is linearly dependent iff $v \in \mathrm{Span}(S)$

Th If $v \in Span(S)$ then $Span(S \cup \{v\}) = Span(S)$

Proof since $v = \lambda_1 v_1 + \cdots \lambda_n v_n$ any linear combination of elements of S and v can be written as a linear combination of elements of S