Lesson 3

Sums, direct sums

Def: If S_1 and S_2 are subspaces of V
the sum $S_1 + S_2$ is the set

$$\{v+w \mid v \in S_1,\ w \in S_2\}$$

Example: $\{(x,0) \mid x \in R\} + \{(0,y) \mid y \in R\} = R^2$

For the rest of these notes if I write
$S_1 + S_2$ I will assume S_1 and S_2 are
subspaces of some vector space V.

Th $S_1 + S_2$ is the smallest subspace of V that contains S_1 and S_2

Proof

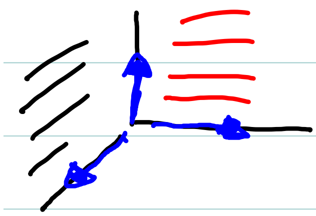
Note: In a similar way we can define the sum of k subsets of V

$S_1 + S_2 + \cdots + S_k = \{ v_1 + v_2 + \cdots + v_k \mid v_i \in S_i \}$

it is still the smallest subset containing $S_1, \dots, S_k$.

Example $S_1 = xz$ plane in R^3 S_2 yz plane in R^3

$S_1 = \{(x\ 0\ z) \mid x, z \in R\}$ $S_2 = \{(0\ y\ z) \mid y, z \in R\}$



$S_1 \cap S_2 = z$ axis

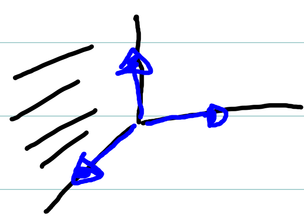
$S_1 + S_2 = R^3$

$(1,1,1) = (1,0,1) + (0,1,0)$ or

$(1,1,1) = (1,0,0) + (0,1,1)$

On the other hand if $S_3 = \{(0\ y\ 0) \mid y \in R\}$ y axis

what is $S_1 + S_3$? $S_1 + S_3 = R^3$



s

$(1\ 1\ 1) = (1\ 0\ 1) + (0\ 1\ 0)$ no other possibilitie

Def The Sum $S_1 + S_2 \cdots + S_n$ is called direct and we shall write $S_1 \oplus S_2 \oplus \cdots \oplus S_n$ if every vector in it can be written in only one way as sum of vectors of S_i.

Th: $U_1 + U_2 \cdots + U_n$ is a direct sum iff $w_1 + w_2 + \cdots + w_n = 0$ with $w_i \in U_i$ implies $w_i = 0$ for $1 \leq i \leq n$

Th: $U_1 + U_2$ is direct sum $\iff U_1 \cap U_2 = \{0\}$

Example: $f\ \mathbb{R} \to \mathbb{R}$ is even if $f(x) = f(-x)$ and odd if $f(-x) = -f(x)$. Let

$E = \{\text{Even functions } \mathbb{R} \to \mathbb{R}\}$ $\quad E \leq \mathbb{R}^{\mathbb{R}}$

$O = \{\text{Odd functions } \mathbb{R} \to \mathbb{R}\}$ $\quad O \leq \mathbb{R}^{\mathbb{R}}$

Proof:

$R^R = E \oplus O$

Proof

Ex Any matrix $M \in M_{n\times n}(\mathbb{R})$ can be written in the form

$$A = \underbrace{\frac{1}{2}(A + A^t)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A^t)}_{\text{Skew symmetric}}$$

$$M_{n\times n}(\mathbb{R}) = \text{Symmetric} \oplus \text{Skew-Symmetric}$$

Note: when considering $U_1 + \cdots + U_n$ it is not sufficient to have $U_i \cap U_j = \{0\}$ to ensure the sum is direct

Ex: $U_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$
$U_2 = \{(0, 0, z) \mid z \in \mathbb{R}\}$
$U_3 = \{(0, y, y) \mid y \in \mathbb{R}\}$

$(0, -1, 0) + (0, 0, -1) + (0, 1, 1) = (0, 0, 0)$
so the sum is not direct but
$U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$

In order for $W_1 + W_2 + \cdots + W_n$ to be a direct sum we want $w_1 + \underbrace{w_2 + w_3 + \cdots + w_n}_{-w_1} = 0 \Rightarrow w_1 = w_2 = \cdots = w_n = 0$

Th: If $W_1, W_2, \cdots W_n$ are subspaces of V then $W_1 + W_2 + \cdots + W_n$ is a direct sum iff $\forall i \le n$

$W_i \cap \sum_{j \neq i} W_j = \{0\}$